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Optimal control of partial differential equations based on the Variational Iteration Method

Abderrahmane Akkouche [a,b], Ahmed Maidi [b,*], Mohamed Aidene [b]

[a] Département des Mathématiques, Faculté des Sciences et des Sciences Appliquées, Université Akli Mohand Oulhadj, 10 000 Bouira, Algérie

[b] Laboratoire de Conception et Conduite des Systèmes de Production, Université Mouloud Mammeri de Tizi-Ouzou, 15 000 Tizi-Ouzou, Algérie

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ABSTRACT

In this work, the Variational Iteration Method is used to solve a quadratic optimal control problem of a system governed by linear partial differential equations. The idea consists in deriving the necessary optimality conditions by applying the minimum principle of Pontryagin, which leads to the well-known Hamilton–Pontryagin equations. These linear partial differential equations constitute a multi-point-boundary value problem. To achieve the solution of the Hamilton–Pontryagin equations using the Variational Iteration Method, an approach is proposed and illustrated by two application examples.

1. Introduction

Unlike the optimal control theory of systems governed by partial differential equations (PDEs) [1–3], optimal control of systems governed by ordinary differential equations (ODEs) is relatively advanced [4,5]. For this kind of system, efficient approaches have been developed (calculus of variations, minimum principle of Pontryagin and dynamic programming), and the solution is achieved in general numerically since the optimality conditions are a set of nonlinear ODEs (Euler–Lagrange equation, Hamilton–Pontryagin equations or Hamilton–Jacobi Bellman equation) [4]. For systems described by PDEs, even though these approaches can be applied, but obtaining the solution, either numerically or analytically, is a difficult task due to the complexity of the calculations to handle [1,3,6–9].

Recently, various methods that provide an approximate analytical of both linear and nonlinear differential equations have been developed in the literature [10]. These methods use practical iterative formulas to provide the solution which may converge to the exact solution if it exists otherwise an approximate analytical solution can be obtained by performing only few iterations. The well-known and established methods are Adomian's Decomposition Method [11], the Homotopy Perturbation Method [12] and the Variational Iteration Method [13].

These methods have been successfully applied to solve optimal control problem of systems governed by ODEs [14–19]. To the best of our knowledge, apart from the contribution of [20], that used the Variational Iteration Method (VIM), these methods are still not applied in the area of optimal control of systems described by PDEs, which motivates this work.

The VIM method developed by [21] is a powerful mathematical tool that provides iteratively the solution of a wide class of both ODEs and PDEs [22–26]. The method provides an approximate analytical solution of differential equations in the

* Corresponding author.
E-mail address: ahmed.maidi@gmail.com (A. Maidi).

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form of an infinite series [13]. The terms of the series are determined using correction functional that involves the Lagrange multiplier [27], as a key element, identified using the calculus of variations theory. In addition, the VIM method starts by considering an initial solution, which is chosen so that the boundary conditions are verified to ensure a rapid convergence [28]. Note that, the convergence of the VIM method has been investigated by many authors [28–31] and the main existing results are developed based on the Banach fixed point theorem [32].

In this paper the VIM method is used to solve the quadratic optimal control problem of systems governed by linear PDEs. The method is applied to achieve the solution of the Hamilton–Pontryagin equations, which constitute the necessary optimality conditions derived using the minimum principle of Pontryagin. Our choice of the VIM method is motivated by the fact that this method needs less effort compared to the Adomian Decomposition Method (ADM) and the Homotopy Perturbation Method (HPM) that need the calculation of the Adomian polynomials, which constitutes a tedious task [33]. In addition, using the VIM method, the solution is achieved using a practical correction functional that can provide either an exact or an approximate analytical solution according to the complexity of the equation. The main advantage of the VIM method over the alternative methods (ADM and HPM) is the existence of sound results concerning its convergence [28,29].

The paper is organized as follows. The linear quadratic optimal control problem is formulated in Section 2. Section 3 summarizes the necessary optimality conditions of the formulated optimal control problem derived using the Pontryagin's minimum principle. In Section 4, the solution of the PDEs using the VIM method is presented. Section 5 is devoted to the application of the VIM method for solving the Hamilton–Pontryagin equations. Application examples are given in Section 6 and the last section is reserved to the conclusion.

2. Linear quadratic optimal control problem

The linear quadratic optimal control problem considered in this work is formulated as follows:

$$\min_{u(z,t)} J(u(t)) = \int_0^{t_f} \int_0^{l} \left[q\left(x^d(z,\ t) - x(z,\ t)\right)^2 + r\, u^2(z,\ t)\right] dz\, dt \tag{1}$$

Subject to

$$x_t(z,\ t) = f\big(x(z,\ t),\ x_z(z,\ t),\ x_{zz}(z,\ t),\ u(z,\ t),\ z,\ t\big), \tag{2}$$

with the initial condition

$$x(z, 0) = x_0(z), \tag{3}$$

and the boundary conditions

$$\alpha_0\, x(0,\ t) + \beta_0\, x_z(0,\ t) = h_0(t), \tag{4}$$

$$\alpha_l\, x(l,\ t) + \beta_l\, x_z(l,\ t) = h_l(t). \tag{5}$$

The final state $x(z,\ t_f)$ can be fixed (specified), that is,

$$x(z,\ t_f) = x_f(z) \tag{6}$$

or free (unspecified) according to the formulation of the optimal control problem.

In the formulated optimal control problem, $x(z,\ t)$ is the state variable assumed to be sufficiently smooth of its arguments z and t, $u(z, t)$ is the control variable, $x_t(z,\ t), x_z(z,\ t)$ are the partial derivatives of $x(z,\ t)$ with respect to t and z, respectively. $x_{zz}(z,\ t)$ is the second partial derivative of $x(z,\ t)$ with respect to z. $z \in [0,\ l]$ and $t \in [0,\ t_f]$ are the spatial and the temporal independent variables, respectively. t_f is the final time assumed to be fixed. $x_0(z)$ is the initial state, $x^d(z,\ t)$ is the desired profile and f is a linear function. q and r are positive weighting factors. α_0, α_l, β_0 and β_l are constants. $h_0(t)$ and $h_l(t)$ are smooth functions.

In the following, it is assumed that $u(z,\ t)$ belongs to the space of admissible controls denoted by $\mathcal{U}$ and the state $x(z,\ t)$ belongs to the space of the reachable state $\mathcal{X}$. In addition, assume that the quadratic function J, defined on $\mathcal{U}$, is continuous and strictly convex, it is also assumed that the set $\mathcal{U}$ is compact, which means that the formulated optimal problem is well-posed and admits a unique solution [34].

The objective is to determine the control variable $u(z,\ t)$ that minimizes the performance index (1) and satisfies the constraints (2)–(6) if the final state is fixed.

3. Necessary optimality conditions

The minimum principle of Pontryagin is an elegant method that allows deriving the necessary optimality conditions for the optimal control problems, that is, the well-known Hamilton–Pontryagin equations and their solution yields the optimal control law. According to the minimum principle of Pontryagin, the solution of the optimal control problem is determined by minimizing the Hamiltonian defined, by introducing the costate variable $p(z,\ t)$, as follows:

$$H(x,\ x_z,\ x_{zz},\ u,\ p,\ z,\ t) = q\left[x^d(z,\ t) - x(z,\ t)\right]^2 + r\, u^2(z,\ t) + p(z,\ t)\, f(x,\ x_z,\ x_{zz},\ u,\ z,\ t), \tag{7}$$

with respect to the control variable $u(z, t)$. Therefore, the expression of the optimal control is determined by solving the following equation

$$\frac{\partial H(x, x_z, x_{zz}, u, p, z, t)}{\partial u} = 0. \tag{8}$$

Eq. (8) gives the expression of the optimal control law as a function of the x and p variables and the partial derivatives x_z and x_{zz}, that is,

$$u(z, t) = \phi(x, x_z, x_{zz}, p, z, t) \tag{9}$$

where ϕ is a linear function.

Therefore, the minimum of the Hamiltonian is

$$H^* = H|_{u(z,t)=\phi(x, x_z, x_{zz}, p, z, t)}. \tag{10}$$

Now, to determine the optimal control law $u^*(z, t)$ as in (9), the optimal solutions $x^*(z, t)$ and costate $p^*(z, t)$ are determined by solving the following set of the state and co-state equations, respectively [2,3]

$$x_t = \frac{\partial H^*}{\partial p} = f(x, x_z, x_{zz}, u, z, t)|_{u=\phi(x, x_z, x_{zz}, p, z, t)} \tag{11}$$

$$p_t = -\left[\frac{\partial H^*}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial H^*}{\partial x_z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial H^*}{\partial x_{zz}}\right)\right] \tag{12}$$

with the transversality boundary conditions

$$\left[\frac{\partial H^*}{\partial x_z} - \frac{\partial}{\partial z}\left(\frac{\partial H^*}{\partial x_{zz}}\right)\right]\delta x + \left(\frac{\partial H^*}{\partial x_{zz}}\right)\delta x_z = 0 \quad \text{both at } z = 0 \text{ and } z = l \tag{13}$$

and the transversality final condition

$$p(z, t)\,\delta x = 0 \quad \text{at } t = t_f \tag{14}$$

where δx and δx_z are the variations of x and x_z, respectively.

In this work, it is proposed to use the VIM method to solve the set of PDEs (11) and (12), known as the Hamilton–Pontryagin equations, with the initial condition (3), the boundary conditions (4)–(5), the transversality conditions (13) and either the final state condition (6) or the transversality final condition (14) according to the nature of the final state, that is, fixed or free. The solution of this multi-point-boundary value problem yields the optimal profiles $x^*(z, t)$ and $p^*(z, t)$ and the optimal control law $u^*(z, t)$ is deduced from (9).

4. Variational Iteration Method

To illustrate the basic idea of the VIM Method, consider the following general PDE, written in an operator form as:

$$L_t y(z, t) + L_z y(z, t) + Ny(z, t) = g(z, t), \tag{15}$$

where L_t and L_z are linear operators of t and z, respectively. N is a non linear operator and g is a given analytical function.

According to the VIM method, a correction functional in the t-direction can be constructed as follows [13,21]:

$$y^{(k+1)}(z, t) = y^{(k)}(z, t) + \int_0^t \lambda(\tau)\left[L_\tau y^{(k)}(z, \tau) + L_z\tilde{y}^{(k)}(z, \tau) + N\tilde{y}^{(k)}(z, \tau) - g(z, \tau)\right]d\tau \tag{16}$$

where λ is the general Lagrange multiplier [27], which can be identified optimally using the calculus of variations theory. $\tilde{y}$ is a restricted variation, that is, $\delta\tilde{y} = 0$. Note that the correction functional in the z-direction can also be used [13,21].

Based on the Banach fixed point theorem, Tatari and Dehghan [29] proved that the sufficient condition that ensures the convergence of the VIM method is the strictly contraction of the operator $\mathcal{A}$ defined as follows:

$$\mathcal{A}y(z, t) = y(z, t) + \int_0^t \lambda(\tau)\ [L_t y(z, \tau) + L_z y(z, \tau) + Ny(z, \tau) - g(z, \tau)]\ d\tau. \tag{17}$$

In this case, the sequence generated by the correction functional (16) converges to the fixed point of the operator $\mathcal{A}$ that represents the solution of the PDE (15). Hence, by choosing a zeroth approximation $y^{(0)}(z, t)$, the successive approximations $y^{(k+1)}(z, t)$, $k \geq 0$, of the solution $y(z, t)$ are determined using the correction functional (16), consequently the solution of (15) is given by

$$y(z, t) = \lim_{k\to\infty} y^{(k)}(z, t). \tag{18}$$

Generally, the exact analytical solution cannot be determined because in practice the calculation of the infinite terms $y^{(k)}(z, t)$ is not possible and for some equations a closed form solution does not exist and can be only approximated. Nevertheless, an approximate analytical solution with a prescribed accuracy ε can be achieved by performing a certain number of iterations denoted in the following by k_ε, that is,

$$y(z, t) \approx y^{(k_\varepsilon)}(z, t). \tag{19}$$

The number of iterations k_ε corresponds to the k that verify

$$\left\| y^{(k)}(z, t) - y^{(k-1)}(z, t) \right\| = \max_{0\le t\le t_f,\, 0\le z\le l} \left| y^{(k)}(z, t) - y^{(k-1)}(z, t) \right| \le \varepsilon \tag{20}$$

where ε is the desired threshold.

5. Solving the Hamilton–Pontryagin equations using VIM

The first step to solve the Hamilton–Pontryagin equations (11) and (12) consists in writing the corresponding correction functionals in the t-direction corresponding as follows:

$$x^{(k+1)}(z, t) = x^{(k)}(z, t) + \int_0^t \lambda_x(\tau) \left\{ x_t^{(k)} - f(x^{(k)}, x_z^{(k)}, x_{zz}^{(k)}, \phi(x^{(k)}, x_z^{(k)}, x_{zz}^{(k)}, p^{(k)}, z, \tau), z, \tau) \right\} d\tau \tag{21}$$

$$p^{(k+1)}(z, t) = p^{(k)}(z, t) + \int_0^t \lambda_p(\tau) \left\{ p_t^{(k)} + \left[\frac{\partial H^*}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial H^*}{\partial x_z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial H^*}{\partial x_{zz}}\right) \right] \right\} d\tau \tag{22}$$

then to start the iteration process, the next step is to select the zeroth approximations of both $x(z, t)$ and $p(z, t)$. Recall that the approximate solutions of the equations (11)–(12) must verify the initial condition (3), the boundary conditions (4)–(5), the transversality boundary conditions (13) and either the final state condition (6) or the transversality final condition (14). Therefore, it is proposed to choose these zeroth approximations as polynomial functions of the independent variable z and t that involve unknown parameters to be identified by imposing both the boundary and the transversality conditions. This choice is justified by the fact that the polynomial functions yield approximate solutions $x^{(k)}(z, t)$ and $p^{(k)}(z, t)$ that are both simple to integrate and to differentiate during the iteration process. The number of the unknown parameters of each zeroth approximation is equal to the number of the boundary and transversality conditions to be verified by the corresponding solution (see the application examples section). Note that, the VIM method converges to the approximate solution whatever the choice made for the zeroth approximations but the number of iterations needed to achieve the solution, with a given accuracy, depends on the chosen zeroth approximations.

Once the zeroth approximations are chosen, the solutions $x^{(k)}(z, t)$ and $p^{(k)}(z, t)$, $k \ge 1$, are determined iteratively, using the correction functionals (21) and (22), and the approximate analytical solutions, with a desired accuracy, are given as follows:

$$x(z, t) \approx x^{(k_\varepsilon)}(z, t), \tag{23}$$

$$p(z, t) \approx p^{(k_\varepsilon)}(z, t). \tag{24}$$

The number of iterations k_ε needed to get the approximate analytical solution is determined based on the performance index (1). Thus, it is proposed to substitute, at each iteration $k \ge 1$, the approximate solution $x^{(k)}(z, t)$ and the control $u^{(k)}(z, t)$ deduced from (9) into the performance index (1) then evaluate its value. The number of iterations k_ε is the k that verifies

$$\left| J\left(u^{(k)}\right) - J\left(u^{(k-1)}\right) \right| \le \varepsilon \tag{25}$$

where ε is the desired threshold.

The proposed design approach to solve the Hamilton–Pontryagin, using the VIM method, can be summarized as:

1. Set $k = 1$ and choose the zeroth approximations $x^{(0)}(z, t)$ and $p^{(0)}(z, t)$ as functions of z and t with unknown parameters as explained above,
2. Determine the approximate solutions $x^{(k)}(z, t)$ and $p^{(k)}(z, t)$ using the correction functionals (21) and (22),
3. Impose the initial, the boundary and the transversality conditions to determine the unknown parameters. If there is no solution for the unknown parameters set $k = k + 1$ and go to step 2, else go to step 4,
4. Substitute the approximate analytical solutions $x^{(k_\varepsilon)}(z, t)$ and $p^{(k_\varepsilon)}(z, t)$ into the optimal control expression (9) and evaluate the performance index (1). If $\left| J\left(u^{(k)}\right) - J\left(u^{(k-1)}\right) \right| \ge \varepsilon$ (ε is a desired threshold), set $k = k + 1$ and go to step 2, else set $k_\varepsilon = k$ and go to step 5,
5. Stop the iteration process; the optimal control law is $u^*(z, t) \approx u^{(k_\varepsilon)}(z, t)$.

6. Application examples

In this section, the proposed approach is illustrated by two application examples. The calculations are performed using the free open-source mathematics software system Sage (http://www.sagemath.org/).

Example 1

Consider the following optimal control problem

$$\min_{u(z,\,t)} J(u(z,\ t)) = \frac{1}{2}\int_0^1 \int_0^1 \left[\left(x^d(z,\ t) - x(z,\ t)\right)^2 + u^2(z,\ t)\right] dz\, dt \tag{26}$$

Subject to:

$$x_t(z,\ t) = x_{zz}(z, t) + u(z, t) + 2\, x^d(z,\ t) \tag{27}$$

$$x(z,\ 0) = 0 \tag{28}$$

$$x(z,\ 1) \text{ is free} \tag{29}$$

$$x(0,\ t) = 0 \tag{30}$$

$$x(1,\ t) = 0 \tag{31}$$

$$x^d(z,\ t) = t^2\, z\,(1 - z). \tag{32}$$

The corresponding Hamiltonian function is

$$H(x,\ x_z,\ x_{zz},\ u,\ p,\ z,\ t) = \frac{1}{2}\left[\left(x^d(z,\ t) - x(z,\ t)\right)^2 + u^2(z,\ t)\right] + p(z,\ t)(x_{zz}(z, t) + u(z, t) + 2\, x^d(z,\ t)). \tag{33}$$

The minimization of the Hamilton function with respect to $u(z,\ t)$ yields

$$\frac{\partial H}{\partial u} = u(z,\ t) + p(z,\ t) = 0, \tag{34}$$

consequently,

$$u(z,\ t) = -p(z,\ t) \tag{35}$$

and the minimum of the Hamilton function is

$$H^*(x,\ x_z,\ x_{zz},\ p,\ z,\ t) = \frac{1}{2}\left[\left(x^d(z,\ t) - x(z,\ t)\right)^2 + p^2(z,\ t)\right] + p(z,\ t)\left(x_{zz}(z,\ t) - p(z,\ t) + 2\, x^d(z,\ t)\right). \tag{36}$$

Thus, in addition to (34), the optimality conditions are

- state and costate equations (Eqs. (11) and (12))

$$x_t(z,\ t) = x_{zz}(z,\ t) - p(z,\ t) + 2\, x^d(z,\ t), \tag{37}$$

$$p_t(z,\ t) = -p_{zz}(z,\ t) - x(z,\ t) + x^d(z,\ t). \tag{38}$$

- transversality boundary conditions (Eq. (13) both at $z = 0$ and $z = 1$)

$$p(0, t) = 0, \tag{39}$$

$$p(1, t) = 0. \tag{40}$$

- transversality final condition (Eq. (14))

$$p(z,\ 1) = 0 \tag{41}$$

since the final state (29) is free.

Now, apply the VIM method to solve these state and costate equations (37)–(38) with the initial condition (28), the boundary conditions (30)–(31) and the transversality conditions (39)–(41), which represent a multi-point-boundary value problem. The corresponding correction functionals in t-direction are:

$$x^{(k+1)}(z,\ t) = x^{(k)}(z, t) + \int_0^t \lambda_x(\tau)\left\{x_\tau^{(k)}(z,\ \tau) - \tilde{x}_{zz}^{(k)}(z,\ \tau) + \tilde{p}^{(k)}(z,\ \tau) - 2\, x^d(z,\ \tau)\right\} d\tau, \tag{42}$$

$$p^{(k+1)}(z, t) = p^{(k)}(z, t) + \int_0^t \lambda_p(\tau)\left\{p_\tau^{(k)}(z,\ \tau) + \tilde{p}_{zz}^{(k)}(z,\ \tau) + \tilde{x}^{(k)}(z,\ \tau) - x^d(z,\ \tau)\right\} d\tau, \tag{43}$$

where λ_x and λ_p are general Lagrange multipliers, and $\tilde{x}^{(k)}(z,\ t)$, $\tilde{x}_{zz}^{(k)}(z,\ t)$, $\tilde{p}^{(k)}(z,\ t)$, $\tilde{p}_{zz}^{(k)}(z,\ t)$ denote the restricted variations, i.e., $\delta\tilde{x}_{zz}^{(k)} = \delta\tilde{x}^{(k)} = \delta\tilde{p}^{(k)} = \delta\tilde{p}_{zz}^{(k)} = 0$.

- *Identification of Lagrange multipliers $\lambda_x(\tau)$ and $\lambda_p(\tau)$.*

  By making the above correction functionals (42) and (43) stationary, we get

$$\delta x^{(k+1)}(z,\ t) = \delta x^{(k)}(z,t) + \delta \int_0^t \lambda_x(\tau) \left\{ x_\tau^{(k)}(z,\ \tau) - \tilde{x}_{zz}^{(k)}(z,\ \tau) + \tilde{p}^{(k)}(z,\ \tau) - 2\,x^d(z,\ \tau) \right\} d\tau, \tag{44}$$

$$\delta p^{(k+1)}(z,t) = \delta p^{(k)}(z,t) + \delta \int_0^t \lambda_p(\tau) \left\{ p_\tau^{(k)}(z,\ \tau) + \tilde{p}_{zz}^{(k)}(z,\ \tau) + \tilde{x}^{(k)}(z,\ \tau) - x^d(z,\ \tau) \right\} d\tau, \tag{45}$$

and the integration by parts yields

$$\begin{aligned} \delta x^{(k+1)}(z,\ t) \ &= \ \delta x^{(k)}(z,t) + \lambda_x(\tau)\,\delta x^{(k)}(z,\tau)\big|_{\tau=t} - \int_0^t \dot{\lambda}_x(\tau)\,\delta x^{(k)}(z,\ \tau)d\tau - \int_0^t \lambda_x(\tau)\,\delta\tilde{x}_{zz}^{(k)}(z,\ \tau)d\tau \\ &\quad + \int_0^t \lambda_x(\tau)\,\delta\tilde{p}^{(k)}(z,\ \tau)\,d\tau - 2\int_0^t \lambda(\tau)\,\delta x^d(z,\ \tau)\,d\tau \end{aligned} \tag{46}$$

$$\begin{aligned} \delta p^{(k+1)}(z,t) \ &= \ \delta p^{(k)}(z,t) + \lambda_p(\tau)\,\delta p^{(k)}(z,\ \tau)\big|_{\tau=t} - \int_0^t \dot{\lambda}_p(\tau)\delta\, p^{(k)}(z,\ \tau)d\tau + \int_0^t \lambda_p(\tau)\,\delta\,\tilde{p}_{zz}^{(k)}(z,\ \tau)\,d\tau \\ &\quad + \int_0^t \lambda_p(\tau)\,\delta\,\tilde{x}^{(k)}(z,\ \tau)d\tau - \int_0^t \lambda_p(\tau)\,\delta x^d(z,\ \tau)\,d\tau, \end{aligned} \tag{47}$$

since $\delta\tilde{x}_{zz}^{(k)} = \delta\tilde{x}^{(k)} = \delta\tilde{p}^{(k)} = \delta\tilde{p}_{zz}^{(k)} = 0$, it follows that

$$\int_0^t \lambda_x(\tau)\,\delta\tilde{x}_{zz}^{(k)}(z,\ \tau)\,d\tau = 0, \tag{48}$$

$$\int_0^t \lambda_x(\tau)\,\delta\tilde{p}^{(k)}(z,\ \tau)\,d\tau = 0, \tag{49}$$

$$\int_0^t \lambda_p(\tau)\,\delta\,\tilde{p}_{zz}^{(k)}(z,\ \tau)\,d\tau = 0, \tag{50}$$

$$\int_0^t \lambda_p(\tau)\,\delta\tilde{x}^{(k)}(z,\ \tau)\,d\tau = 0, \tag{51}$$

$$\int_0^t \lambda_p(\tau)\,\delta x^d(z,\ \tau)\,d\tau = 0, \tag{52}$$

$$\int_0^t \lambda(\tau)\,\delta x^d(z,\ \tau)\,d\tau = 0, \tag{53}$$

therefore the stationary conditions are

$$\begin{aligned} &1 + \lambda_x(\tau = t) = 0, \\ &\dot{\lambda}_x(\tau) = 0, \\ &1 + \lambda_p(\tau = t) = 0, \\ &\dot{\lambda}_p(\tau) = 0, \end{aligned} \tag{54}$$

which yields

$$\lambda_x(\tau) = \lambda_p(\tau) = -1. \tag{55}$$

By substituting the value of λ_x and λ_p into the functionals (42)–(43), we obtain:

$$x^{(k+1)}(z,\ t) = x^{(k)}(z,t) - \int_0^t \left\{ x_\tau^{(k)}(z,\ \tau) - x_{zz}^{(k)}(z,\ \tau) + p^{(k)}(z,\ \tau) - 2\,x^d(z,\ \tau) \right\} d\tau, \tag{56}$$

$$p^{(k+1)}(z,t) = p^{(k)}(z,t) - \int_0^t \left\{ p_\tau^{(k)}(z,\ \tau) + p_{zz}^{(k)}(z,\ \tau) + x^{(k)}(z,\ \tau) - x^d(z,\ \tau) \right\} d\tau. \tag{57}$$

- *Selection of the zeroth approximations $x^{(0)}(z,t)$ and $p^{(0)}(z,t)$.*

  The solution $x(z,\ t)$ must satisfy the initial condition (28), the two boundary conditions (30) and (31) while the solution $p(z,\ t)$ must satisfy the three transversality conditions (39)–(41). Thus, both the zeroth approximations $x^{(0)}(z,\ t)$ and $p^{(0)}(z,\ t)$ are chosen as polynomial functions that involve three unknown parameters given as follows:

$$x^{(0)}(z,\ t) = a_0\,z + a_1\,t + a_2 \tag{58}$$

$$p^{(0)}(z,\ t) = b_0\,z + b_1\,t + b_2 \tag{59}$$

where a_i and b_i ($i = 0, 1, 2$) are unknown parameters to be determined by imposing the initial, the boundary and transversality conditions as explained above.

- *Determination of the approximate solutions of $x(z, t)$ and $p(z, t)$.*

To illustrate the proposed approach, to solve the optimality conditions using the VIM method, the two first iterations are discussed at length in the following.

Thus, the iteration formulas (56) and (57) give for $k = 1$

$$x^{(1)}(z, t) = a_0 z + a_2 - b_0 z t - \frac{b_1 t^2}{2} - b_2 t + \frac{2 t^3 z (1 - z)}{3} \tag{60}$$

$$p^{(1)}(z, t) = b_0 z + b_2 - a_0 z t - \frac{a_1 t^2}{2} - a_2 t + \frac{t^3 z (1 - z)}{3}. \tag{61}$$

By imposing the initial condition (28), the boundary conditions (30)–(31) and the transversality boundary conditions (39)–(41), that is,

$$x^{(1)}(z, 0) = 0 \tag{62}$$

$$x^{(1)}(0, t) = 0 \tag{63}$$

$$x^{(1)}(1, t) = 0 \tag{64}$$

$$p^{(1)}(0, t) = 0 \tag{65}$$

$$p^{(1)}(1, t) = 0 \tag{66}$$

$$p^{(1)}(z, 1) = 0 \tag{67}$$

it follows that

$$a_0 = 0, \qquad a_1 = \frac{2 z (z - 1)}{3 (t^2 - 1)}, \qquad a_2 = 0, \qquad b_0 = 0, \qquad b_1 = -\frac{2 t z (z - 1)}{3 (t^2 - 1)}, \qquad b_2 = \frac{z (z - 1) t^2}{3 (t^2 - 1)}. \tag{68}$$

Thus the first approximate solutions are

$$x^{(1)}(z, t) = \frac{2 t^3 z (1 - z)}{3} \tag{69}$$

$$p^{(1)}(z, t) = \frac{t^3 z (1 - z)}{3} \tag{70}$$

and the first approximate solution of the optimal control is

$$u^{(1)}(z, t) = -\frac{t^3 z (1 - z)}{3}. \tag{71}$$

The evaluation of the performance index (26) yields

$$J\left(u^{(1)}\right) = \frac{1}{2} \int_0^1 \int_0^1 \left[\left(x^d(z, t) - x^{(1)}(z, t)\right)^2 + \left(u^{(1)}(z, t)\right)^2\right] dz\, dt = 0.00095238. \tag{72}$$

The results obtained for the rest of the iterations are summarized in Table 1. By assuming a threshold $\varepsilon = 10^{-7}$, the VIM method converges after 6 iterations and leads to the following approximate analytical optimal control

$$u^*(z, t) \approx u^{(6)}(z, t) = -\frac{\left(t^5 - 4 t^4 + 56 t^3 - 168 t^2 + 1680 t - 3360\right) t^3 z\, (z - 1)}{10080} \tag{73}$$

and the optimal state and costate are

$$x^*(z, t) \approx x^{(6)}(z, t) = \frac{(t^5 - 16 t^4 + 56 t^3 - 672 t^2 + 1680 t - 13440)\, t^3 z\, (z - 1)}{20160} \tag{74}$$

$$p^*(z, t) \approx p^{(6)}(z, t) = \frac{\left(t^5 - 4 t^4 + 56 t^3 - 168 t^2 + 1680 t - 3360\right) t^3 z\, (z - 1)}{10080}. \tag{75}$$

Example 2.

Let us consider the following optimal control problem

$$\min_{u(z,t)} J(u(z, t)) = \frac{1}{2} \int_0^{0.5} \int_0^1 \left[5\, (x^d(z, t) - x(z, t))^2 + u^2(z, t)\right] dz\, dt \tag{76}$$

Table 1
Iteration results.

Iteration k	$\left\|J\left(u^{(k)}\right)-J\left(u^{(k-1)}\right)\right\|$
1	–
2	0.183715461e−5
3	0.524223719e−4
4	0.984740568e−6
5	0.933596337e−6
6	0.130975053e−7
7	0.105275454e−7
8	0.110904750e−9

Subject to:

$$x_t(z,\ t) = z\,(z-1)\,x_{zz}(z,t) + u(z,t) \tag{77}$$

$$x(z,\ 0) = 0 \tag{78}$$

$$x(z,\ 0.5) \text{ is free} \tag{79}$$

$$x(0,\ t) = t^2 \tag{80}$$

$$x(1,\ t) = 0 \tag{81}$$

$$x^d(z,\ t) = t^2\,(1-z). \tag{82}$$

The corresponding Hamiltonian function is

$$H(x,\ x_z,\ x_{zz},\ u,\ p,\ z,\ t) = \frac{1}{2}\Big[5\,(x^d(z,\ t) - x(z,\ t))^2 + u^2(z,\ t)\Big] + p(z,\ t)\big[z\,(z-1)\,x_{zz}(z,t) + u(z,t)\big]. \tag{83}$$

The expression of the optimal control is

$$u(z,\ t) = -p(z,\ t) \tag{84}$$

and the optimality conditions are

$$x_t(z,\ t) = z\,(z-1)\,x_{zz}(z,\ t) - p(z,\ t) \tag{85}$$

$$p_t(z,\ t) = -\left[5\left(x(z,\ t) - x^d(z,\ t)\right) + \frac{\partial^2\big(z\,(z-1)\,p(z,\ t)\big)}{\partial z^2}\right] \tag{86}$$

$$z\,(z-1)\,p(z,\ t) = 0 \quad \text{for } z = 0 \tag{87}$$

$$z\,(z-1)\,p(z,\ t) = 0 \quad \text{for } z = 1 \tag{88}$$

$$p(z,\ 0.5) = 0. \tag{89}$$

The corresponding correction functionals are given as follows:

$$x^{(k+1)}(z,\ t) = x^{(k)}(z,t) - \int_0^t \left\{x_\tau^{(k)}(z,\ \tau) - z\,(z-1)\,x_{zz}^{(k)}(z,\ \tau) + p^{(k)}(z,\ \tau)\right\} d\tau, \tag{90}$$

$$p^{(k+1)}(z,t) = p^{(k)}(z,t) - \int_0^t \left\{p_\tau^{(k)}(z,\ \tau) + \frac{\partial^2\big(z\,(z-1)\,p^{(k)(z,\ \tau)}\big)}{\partial z^2} + 5\,x_z^{(k)}(z,\ \tau) - 5\,x^d(z,\ \tau)\right\} d\tau. \tag{91}$$

The solution $x(z,\ t)$ must satisfy the initial condition (78), the boundary conditions (80)–(81), thus the zeroth approximation $x^{(0)}(z,\ t)$ is chosen as a polynomial function with three unknown parameters. The solution $p(z,\ t)$ must satisfy only the terminal condition (89) since the transversality boundary conditions (87)–(88) are checked, therefore the zeroth approximation $p^{(0)}(z,\ t)$ is chosen as polynomial function with a single unknown parameter. The zeroth approximations are chosen as follows:

$$x^{(0)}(z,\ t) = a_0\,z + a_1\,t + a_2 \tag{92}$$

$$p^{(0)}(z,\ t) = b_0\,t. \tag{93}$$

The results of the iteration process are summarized in Table 2. The optimal control is determined by assuming a threshold $\varepsilon = 10^{-5}$. The obtained results show that the VIM method converges after 3 iterations. The optimal state, costate and control are not reported here due to their long expressions. The 3D profiles of the state, costate and control are given in Figs. 1–3, respectively.

Table 2
Iteration results.

Iteration k	$\left\|J\left(u^{(k)}\right)-J\left(u^{(k-1)}\right)\right\|$
1	–
2	0.126727687e−4
3	0.421431732e−5

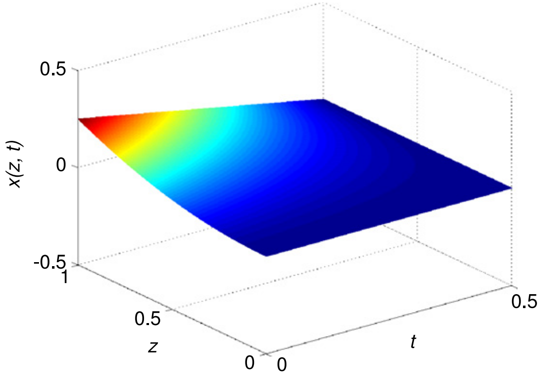

Fig. 1. Optimal state profile $x^*(z, t)$.

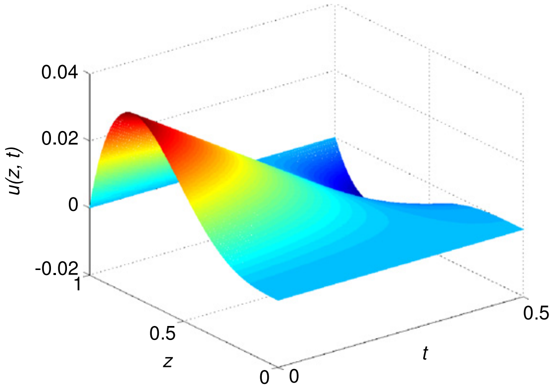

Fig. 2. Optimal costate profile $p^*(z, t)$.

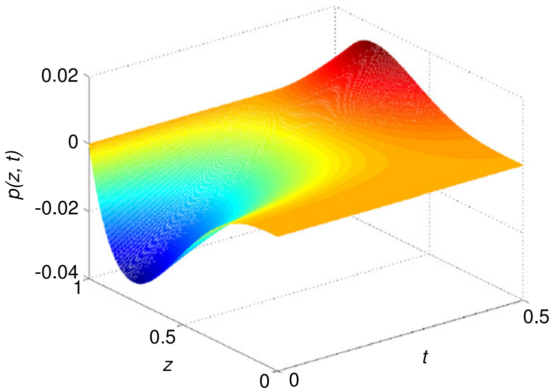

Fig. 3. Optimal control profile $u^*(z, t)$.

7. Conclusion

In this paper, the VIM method is adopted to solve the optimality conditions of the linear quadratic optimal control problem of systems governed by PDEs. These optimality conditions are given by the well-known Hamilton–Pontryagin equations, derived based on the minimum principle of Pontryagin, with appropriate boundary and transversality conditions. Since the system is linear and the performance index is quadratic, the Hamilton–Pontryagin equations are given as a set of coupled linear PDEs. To solve these equations, using the VIM method, an approach is proposed. The approximate analytical solutions are determined iteratively using a correction functional starting by polynomial functions as zeroth approximations of both state and co-state equations with unknown parameters that are determined by imposing both the boundary and transversality conditions. The proposed approach is illustrated by two application examples.

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