Chapter 3

An Extension of Massera's Theorem for *N*-Dimensional Stochastic Differential Equations

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we consider a periodic SDE in the dimension $n \geq 2$, and we study the existence of periodic solutions for this type of equations using the Massera principle. On the other hand, we prove an analogous result of the Massera's theorem for the SDE considered.

Keywords: stochastic differential equations, periodic solution, Markov process, Massera theorem

1. Introduction

The theory of stochastic differential equations is given for the first time by Itô [7] in 1942. This theory is based on the concept of stochastic integrals, a new notion of integral generalizing the Lebesgue–Stieltjes one.

The stochastic differential equations (SDE) are applied for the first time in the problems of Kolmogorov of determining of Markov processes [8]. This type of equations was, from the first work of Itô, the subject of several investigations; the most recent include the generalization of known results for EDO, such as the existence of periodic and almost periodic solutions. It has, among others, the work of Bezandry and Diagana [1, 2], Dorogovtsev [4], Vârsan [12], Da Prato [3], and Morozan and his collaborators [10, 11].

The existence of periodic solutions for differential equations has received a particular interest. We quote the famous results of Massera [9]. In its approach, Massera was the first to establish a relation between the existence of bounded solutions and that of a periodic solution for a nonlinear ODE.

In this work, we will prove an extension of Massera's theorem for the following:

nonlinear SDE in dimension $n \geq 2$

$$dx = a(t,x)dt + b(t,x)dW_t$$

2. Preliminaries

Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be the complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions

- $\{F_t\}_{t\geq 0}$ is an increasing family of sub algebras containing negligible sets of F and is continuous at right.

$$F_\infty = \sigma\{\cup_{t\geq 0} F_t\}.$$

Let a Brownian motion $W(t)$, adapted to $\{F_t, t\geq 0\}$, i.e., $W(0) = 0, \forall t \geq 0, W(t)$ is F_t-measurable. We consider the SDE

$$\begin{cases} dx = a(t,x)dt + b(t,x)dW_t \\ x(t_0) = z. \end{cases} \tag{1}$$

in $(\Omega, F, \{F_t\}_{t\geq 0}, P)$.

The functions $a(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $b(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable. We suppose that F_t is the completion of $\sigma\{W_r, t_0 \leq r \leq t\}$ for all $t \geq t_0$, and the initial condition z is independent of W_t, for $t \geq t_0$ and $E|z|^p < \infty$.

Suppose that the functions $a(t,x)$ and $b(t,x)$ satisfy the global Lipschitz and the linear growth conditions

$$\exists k > 0, \forall t \in \mathbb{R}_+, \forall x, y \in \mathbb{R}^n : \|a(t,x) - a(t,y)\| + \|b(t,x) - b(t,y)\| \leq k\|x - y\|$$

and

$$\|a(t,x)\|^p + \|b(t,x)\|^p \leq k^p(1 + \|x\|^p)$$

We know that if a and b satisfy these conditions, then the system (1) admits a single global solution.

We note by B the space of random F_t-measurable functions $x(t)$ for all t, satisfying the relation

$$\sup_{t\geq 0} E|x(t)|^2,$$

we consider in B the norm

$$\|x\|_B = \sup_{t\geq 0}\left(E|x|^2\right)^{\frac{1}{2}}$$

$\left(B, \|.\|_B\right)$ is the Banach space.

2.1. Markov property

The following result proves that the solution of the SDE (1) is a Markov process.

Theorem 1. ([5], Th. 2, p. 466) *Assume that $a(t,x)$ and $b(t,x)$ satisfy the hypothesis of the theorem ([5], Th. 1, p. 461) and that $X^{(t,x)}(s)$ is a process such that for $s \in t, \infty)$ for all $t > t_0$ is a solution of SDE*

$$X^{(t,x)}(s) = x + \int_t^s a\left(u, X^{(t,x)}(u)\right)du + \int_t^s b\left(u, X^{(t,x)}(u)\right)dW_u \tag{2}$$

Then the process X_t, solution of SDE (1), is a Markovian process with a transition function

$$p(t,x;s,A) = P\left(X^{(t,x)}(s) \in A\right).$$

Let $p(s,x;t,A)$ be a transition function; we construct a Markov process with an initial arbitrary distribution. In a particular case, for $t > s$, we associate with the function $p(s,x;t,A)$ a family $X^{(s,z)}(t,\omega)$ of a Markov process such that the processes $X^{(s,z)}(t,\omega)$ exist with initial point z in s, i.e.,

$$P\left(X^{(s,z)}(t,\omega) = z\right) = 1 \tag{3}$$

2.2. Notions of periodicity and boundedness

Définition 1. *A stochastic process $X(t,\omega)$ is said to be periodic with period T $(T > 0)$ if its finite dimensional distributions are periodic with periodic T, i.e., for all $m \geq 0$, and $t_1, t_2, \ldots t_m \in \mathbb{R}^+$ the joint distributions of the stochastic processes $X_{t_1+kT}(\omega), X_{t_2+kT}(\omega), \ldots X_{t_m+kT}(\omega)$ are independent of k $(k \in \mathbb{Z})$.*

Remark 1. *If $X(t,\omega)$ is $T-$periodic, then $m(t) = EX(t)$, $v(t) = VarX(t)$ are $T-$periodic, in this case, this process is said to be $T-$periodic in the wide sense.*

Définition 2. *The function $p(s,x;t,A) = P(X_t \in A/X_s)$ for $0 \leq s \leq t$, is said to be periodic if $p(s,x;t+s,A)$ is periodic in s.*

Définition 3. *The Markov families* $X^{(t_0,z)}(\omega)$ *are said to be* $p-$*uniformly bounded* $(p > 2)$, *if* $\forall \alpha > 0, \exists \theta(\alpha) > 0, \forall t \geq t_0$:

$$\|z\|_{B,p} \leq \alpha \Rightarrow \left\|X^{(t_0,z)}(\omega)\right\|_{B,p} \leq \theta(\alpha)$$

We denote $X^{(t_0,z)}(\omega)$ as the family of all Markov process for $t_0 \in \mathbb{R}^+$and z in L^p.

Remark 2. *It is easy to see that all* L_p-*borné Markov processes* X_t, $\left(i.e \exists M > 0, \forall t \geq t_0 : \|X_t\|_{B,p}^p \leq M,\right)$ *is* $p-$*uniformly bounded.*

Lemme 1. ([6], Theorem 3.2 and Remark 3.1, pp. 66–67) *A necessary and sufficient condition for the existence of a Markov* $T-$*periodic* $X^{(t_0,z)}(\omega)$ *with a given* $T-$*periodic transition function* $p(s,x;t,A)$, *is that for some* t_0, z, $X^{(t_0,z)}(\omega)$ *are uniformly stochastically continuous and*

$$\lim_{R\to\infty} \lim_{L\to\infty} \inf \frac{1}{L} \int_{t_0}^{t_0+L} p\left(t_0, z; t, \overline{U}_{R,p}\right) dt = 0 \tag{4}$$

if the transition function $p(s, X_s; t, A)$ *satisfies the following not very restrictive assumption*

$$\alpha(R) = \sup_{z \in U_{\beta(R),p}} 0 < t_0, t - t_0 < Tp\left(t_0, z; t, \overline{U}_{R,p}\right) \to_{R\to\infty} 0 \tag{5}$$

for some function $\beta(R)$ *which tends to infinity as* $R \to \infty$.

In Eq. (4), we have $R \in \mathbb{R}_+^*$:

$$U_{R,p} = \{x \in \mathbb{R}^n : |x|^p < R\}$$

$$\overline{U}_{R,p} = \{x \in \mathbb{R}^n : |x|^p \geq R\}$$

The conditions of Lemma 1 are of little use for stochastic differential equations, since the properties of transition functions of such processes are usually not expressible in terms of the coefficients of the equation. So, in the following, we will give some new useful sufficient conditions in terms of uniform boundedness and point dissipativity of systems.

Lemme 2. *If Markov families* $X^{(t_0,z)}(\omega)$ *with* $T-$*periodic transition functions are uniformly bounded uniformly stochastically continuous, then there is a* $T-$*periodic Markov process.*

Proof. By using a Markov inequality [13], we have

$$p\left(t_0, z; t, \overline{U}_{R,p}\right) = \frac{1}{RP(X_{t_0} = z)} E\left|X^{(t_0,z)}(\omega)\right|^p$$
$$\leq \frac{1}{RP(z)} \left\|X^{(t_0,z)}(\omega)\right\|_{B,p}^p$$

Then, for $\alpha > 0, \exists \theta(\alpha) > 0$, such that for all $t \geq t_0$

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$$\|z\|_{B,p} \leq \alpha \Rightarrow \left\|X^{(t_0,z)}(\omega)\right\|_{B,p} \leq \theta(\alpha)$$

we get

$$p\left(t_0, z; t, \overline{U}_{R,p}\right) \leq \frac{1}{RP(z)} \theta^p(\alpha)$$

Then

$$0 \leq \lim_{R\to\infty} \lim_{L\to\infty} \inf \frac{1}{L} \int_{t_0}^{t_0+L} p\left(t_0, z; t, \overline{U}_{R,p}\right) dt \leq \lim_{R\to\infty} \frac{1}{RP(z)} \theta^p(\alpha) \left(\lim_{L\to\infty} \inf \frac{1}{L} \int_{t_0}^{t_0+L} dt\right)$$
$$= \lim_{R\to\infty} \frac{\theta^p(\alpha)}{RP(z)} = 0,$$

that is, Eq. (4). From Lemma 1, we have a $T-$periodic Markov process.

3. Main result

Let the SDE

$$\begin{cases} dx = a(t,x)dt + b(t,x)dW_t \\ x_{t_0} = z, \quad E|z|^p < \infty \end{cases} \tag{6}$$

We assume that this SDE satisfies the conditions as in Section 2 after Eq. (1).

Suppose that

H_1) the functions $a(t,x)$ and $b(t,x)$ are $T-$periodic in t.

H_2) the functions $a(t,x)$ and $b(t,x)$ satisfy the condition

$$\|a(t,x)\|^p + \|b(t,x)\|^p \leq \phi(\|x\|^p), p > 2 \tag{7}$$

where ϕ is a concave non-decreasing function.

Lemme 3. ([13], Lemme 3.4) *Assume that* $a(t,x)$*and* $b(t,x)$ *verify*

$$E(\|a(t,x)\|^p) + E(\|b(t,x)\|^p) \leq \eta, p > 2$$

then, the solutions of periodic SDE (6) are uniformly stochastically continuous.

We prove the Massera's theorem for the SDE in dimension $n \geq 2$.

Theorem 2. *Under* $(H_1), (H_2)$, *if the solutions of the SDE (6) are* L_p-*bounded, then there is a* $T-$*periodic Markov process.*

Proof. We note by $X^{(t_0,z)}(t,\omega)$ an L_p-bounded solution of SDE (6), from Theorem 1, this solution is unique a Markov process that is F_t-measurable. Suppose that $p(t_0,z;t,A)$ is a transition function of Markov process $X^{(t_0,z)}(t,\omega)$, under (H_1) and since $p(t_0,z;t,A)$ depend of $a(t,x), b(t,x)$ then this function is $T-$periodic in t. In the other hand, ϕ is concave non-decreasing function, we get

$$E\phi(|x|^p) \leq \phi(E|x|^p)$$

From the L_p-boundedness of $X^{(t_0,z)}(t,\omega)$, then under (H_2): $\exists\eta > 0$ such that

$$E\left\|a\left(t, X^{(t_0,z)}(t,\omega)\right)\right\|^p + E\left\|b\left(t, X^{(t_0,z)}(t,\omega)\right)\right\|^p < \eta$$

for $p > 2$. By Lemma 3, we have $X^{(t_0,z)}(t,\omega)$ is $p-$uniformly bounded and $p-$uniformly stochastically continuous, this gives, the conditions of Lemma 2 are verified, finally, we can conclude the existence of the $T-$periodic Markov process. □

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