UDC 517.4

Boudref Mohamed-Ahmed

Titchmarsh's theorem of Hankel transform

Hankel transform (or Fourier-Bessel transform) is a fundamental tool in many areas of mathematics and engineering, including analysis, partial differential equations, probability, analytic number theory, data analysis, etc. In this article, we prove an analog of Titchmarsh's theorem for the Hankel transform of functions satisfying the Hankel-Lipschitz condition.

Bibliography: 11 items.

Преобразование Ханкеля (или преобразование Фурье-Бесселя) является фундаментальным инструментом во многих областях математики и техники, включая анализ, уравнения в частных производных, вероятности, аналитическую теорию чисел, анализ данных и т.д. В этой статье мы докажем аналог теоремы Титчмарша для преобразования функций Ханкеля, удовлетворяющих условию Ханкеля-Липшица.

Библиография: 11 названий.

Keywords: Hankel transform, Titchmarsh theorem, Generalized derivatives in the sense of Levi.

Ключевые слова: преобразование Ханкеля, теорема Титчмарша, обобщенные производные в смысле Леви.

1. Introduction

Integral transformations are widely used in mathematics and physics. In fact, methods associated with these transformations have found a vast field of application in recent times, such as data analysis and processing. We distinguish several types of integral transformations [11], and the most commonly cited in literature are:

- Fourier, Laplace and Mellin transforms [2, 4].

- Bessel Integral transformation [5], or more exactly the Hankel,Meijer, Kontorovitch-Lebedev integral transformations [3,6].

For reasons of application, any new result in this area is of theoretical and applied interest. Among these results is the generalization of Titchmarsh's theorem [10]. Titchmarsh has given a relation between a function $f(x)$ and its Fourier transformation $\hat{f}(u)$, where it stated in the following theorem:

THEOREM 1. *[10] Let $\alpha \in (0,1)$, and assume that $f \in L^2(\mathbb{R})$. Then the following are eguivalents*

1.

$$\|f(x+h)-f(x)\|_{L^2} = O\left(h^{\alpha}\right) \quad \text{as } h \to 0$$

2.

$$\int_{|u|\geq N} \left|\hat{f}(u)\right|^2 du = O\left(N^{-2\alpha}\right),\ N \to \infty,$$

where $\hat{f}$ stands for the Fourier transform of f.

Later, this Titchmarsh's theorem had been generalized for some integral transformations, namely:

- the Dunkel transformation [7],
- the Jacobi transformation [1],
- the Fourier-Walsh transformation [9].

2. Hankel transform

Let $f(x) \in L^2(\mathbb{R}_+)$, the Fourier-Bessel transform of order p is defined by:

$$g(u) = F_p[f(x)] = \int_0^{\infty} \sqrt{xu} J_p(xu) f(x) dx, \quad u > 0, p > -\frac{1}{2}, \tag{1}$$

where $J_p(x)$ is the Bessel function of the first kind of order p. The inverse Fourier-Bessel transform is given by:

$$f(x) = F_p^{-1}[g(u)] = \int_0^{\infty} \sqrt{xu} J_p(xu) g(u) du. \tag{2}$$

We call (1) Hankel transform of order p, we can write:

$$H_p\left[f(x)\right](u) = \int_0^\infty \sqrt{xu} J_p(xu) g(u) du, \quad u > 0, p > -\frac{1}{2}.$$

For $f \in L^1\left((0,\infty)x^{2\mu+1}\right)$ we can give another form of the transformation of Henkel:

$$H_\mu\left[f(x)\right](u) = \int_0^\infty (xu)^\mu J_\mu(xu)\, x^{2\mu+1} dx, \quad x \in \mathbb{R}_+,$$

the inverse formula is given by:

$$f(x) = \int_0^\infty (xu)^{\frac{1}{2}} J_p(xu) du \int_0^\infty (ut)^{\frac{1}{2}} J_p(ut) f(t) dt. \tag{3}$$

THEOREM 2. *[10] Let $f(x)$ be a function with bounded variation over any finite interval $(0,R)$ and*

$$\int_0^\infty |f(x)|\sqrt{x} dx < \infty.$$

If $p > -\frac{1}{2}$, then:

$$\frac{1}{2}\left\{f(x+0) + f(x-0)\right\} = \int_0^\infty (xu)^{\frac{1}{2}} J_p(xu) du \int_0^\infty (ut)^{\frac{1}{2}} J_p(ut) f(t) dt.$$

At the points of continuity, we have the formula (3).

Using trigonometric functions, we can write the Hankel transformation of orders $\frac{1}{2}$ and $-\frac{1}{2}$ as follows:

$$F_s(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin ut dt,$$
$$F_c(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ut dt,$$

since

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \sin x, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \cos x.$$

We consider the normed space $L^2(\mathbb{R}_+)$ of functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ whose norm is given by:

$$\|f\|_{L^2(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

THEOREM 3. *If $f(x)x^{p+\frac{1}{2}} \in L^2(\mathbb{R}_+)$, then the Hankel transform $F_p[f](u)u^{p+\frac{1}{2}} \in L^2(\mathbb{R}_+)$, where $g(u) = F_p[f](u)$ is the Hankel transform of $f(x)$. And we have the Parseval's identity:*

$$\int_0^\infty |g(u)|^2 u^{2p+1} du = 2^{2p}\Gamma(p+1) \int_0^\infty |f(x)|^2 x^{2p+1} dx.$$

As a special case, if $f(x) \in L^2(\mathbb{R})$, then we have:

$$\int_0^\infty |g(u)|^2 du = \int_0^\infty |f(x)|^2 dx.$$

We define the normalized Bessel function of the first kind by:

$$j_p(\sqrt{\lambda}x) = \frac{2^p\Gamma(p+1)}{\left(\sqrt{\lambda}x\right)^p} J_p\left(\sqrt{\lambda}x\right).$$

This function $j_p(\sqrt{\lambda}x)$ is the solution of the Bessel equation

$$y'' + \frac{2p+1}{x}y' + \lambda y = 0,$$

with

$$y(0) = 1, y'(0) = 0.$$

For $x^{p+\frac{1}{2}} f(x) \in L^2(\mathbb{R}_+)$ the Hankel transform is given by:

$$g(u) = \frac{u^{p+\frac{1}{2}}}{2^p\Gamma(p+1)} \int_0^\infty j_p(xu) f(x) x^{2p+1} dx,$$

and the inverse formula is:

$$x^{p+\frac{1}{p}} f(x) = \frac{x^{p+\frac{1}{2}}}{2^{2p}\Gamma^2(p+1)} \int_0^\infty j_p(xu) g(u) u^{2p+1} du.$$

The Parseval's identity is given by

$$\int_0^\infty |g(u)|^2 u^{2p+1} du = 2^{2p}\Gamma(p+1) \int_0^\infty |f(x)|^2 x^{2p+1} dx.$$

3. Construction of the $W_2^{r,k}(D)$pace. Modulus of continuity

In $L^2(\mathbb{R}_+)$ let the operator T_h be defined as

$$T_h[f(x)] = \frac{\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}\int_0^\pi f\left(\sqrt{x^2+h^2-2xh\cos t}\right)\sin^{2p} t dt,$$

which corresponds to the following Bessel differential operator:

$$D = \frac{d^2}{dx^2} + \frac{2p+1}{x}\frac{d}{dx}.$$

It is easy to see that

$$T_0[f(x)] = f(x).$$

If $f \in C^1$, then

$$\left.\frac{\partial}{\partial h}\right|_{h=0} T_h[f(x)] = 0.$$

If $f \in C^2$, then $T_h[f(x)] = u(x,h)$ is the solution of the Cauchy problem

$$\begin{cases} u''_{xx} + \frac{2p+1}{x}u'_x = u''_{hh} + \frac{2p+1}{x}u'_h \\ u(x,0) = f(x), u'_h(x,0) = 0. \end{cases}$$

We have the following properties of T_h :

- T_h is continuous, linear and homogeneous.
- $T_h(j_p(\lambda x)) = j_p(\lambda h)j_p(\lambda x)$, for $\lambda > 0$.
- T_h is self-adjoint. If $f(x)$ is continuous, such that

$$\int_0^\infty x^{2p+1}|f(x)|dx < \infty$$

  and $g(x)$ is continuous and bounded for all $x \geq 0$, then:

$$\int_0^\infty T_h[f(x)]\, g(x)x^{2p+1}dx = \int_0^\infty f(x)T_h[g(x)]\, x^{2p+1}dx.$$

- $\|T_h[f] - f\| \to 0$, as $h \to 0$.

We define the first and higher order finite differences of $f(x)$ as follows:

$$\Delta_h f(x) = T_h[f(x)] - f(x),$$

and

$$\begin{aligned}\Delta_h^k f(x) &= \Delta\left(\Delta_h^{k-1} f(x)\right) \\ &= \sum_{i=0}^{k} (-1)^{k-i} C_i^k T_h^i [f(x)].\end{aligned}$$

DEFINITION 1. [8] We define the $k-$order generalized modulus of continuity of function $f(x) \in L^2(\mathbb{R}_+)$ by:

$$\omega^k(f,\delta) = \sup_{0<h\leq\delta} \left\|\Delta_h^k f(x)\right\|_{L^2(\mathbb{R}_+)}, \quad k \geq 1.$$

This modulus of continuity checked the following properties:

1. $\omega(f,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$.
2. $\omega(f,\delta) \nearrow$.
3. $\forall n \in \mathbb{N} : \omega(nf,\delta) \leq n\omega(f,\delta)$.
4. $\forall(\delta_1,\delta_2) \in \mathbb{R}_+ : \omega(f,\delta_1+\delta_2) \leq \omega(f,\delta_1) + \omega(f,\delta_2)$.
5. *If* $f(x) \in Lip(\alpha)$ *then* $\omega(f,\delta) = O(\delta^\alpha)$.
6. $\forall\lambda \in \mathbb{R} : \omega(\lambda f,\delta) \leq (\lambda+1)\omega(f,\delta)$.

Let $W_2^{r,k}(D)$ be the class of functions $f \in L^2(\mathbb{R}_+)$ that have generated derivatives in the sense of Levi satisfying

$$\omega^k(D^r f,\delta) = O\left(\delta^k\right).$$

where D is the Bessel operator defined before.

We have:

$$T_h[f(x)] = \frac{1}{2^{2p}\Gamma^2(p+1)} \int_0^\infty j_p(uh)\, j_p(ux)\, g(u) u^{2p+1} du,$$

then:

$$T_h\left[f(x)\right]-f(x)=\frac{1}{2^{2p}\Gamma^2\left(p+1\right)}\int_0^\infty\left(j_p\left(uh\right)-1\right)j_p\left(ux\right)g(u)u^{2p+1}du.$$

Combining with the Parseval's identity, we get:

$$\left\|T_h\left[f(x)\right]-f(x)\right\|^2=\int_0^\infty\left(j_p\left(uh\right)-1\right)^2\left|g(u)\right|^2u^{2p+1}du,$$

for any $f\in W_2^{r,k}\left(D\right)$, we have

$$\left\|\Delta_h^kD^rf(x)\right\|^2_{L^2\left(\mathbb{R}_+,W_2^{r,k}(D)\right)}=\int_0^\infty\left(j_p\left(uh\right)-1\right)^{2k}\left|g(u)\right|^2u^{4r+2p+1}du.$$

4. Main results

We consider the function $f(x)\in L^2\left(\mathbb{R}_+,W_2^{r,k}\left(D\right)\right)$ with $k=1$, such that:

$$\omega\left(D^rf,\delta\right)=O\left(\delta\right).$$

And we have:

$$\left\|\Delta_hD^rf(x)\right\|^2_{L^2(\mathbb{R}_+,W_2^r(D))}=\int_0^\infty\left(j_p\left(uh\right)-1\right)^2\left|g(u)\right|^2u^{4r+2p+1}du.$$

DEFINITION 2. Let $f(x)\in L^2\left(\mathbb{R}_+,W_2^r\left(D\right)\right)$, we say that f is α-Lipschitzian, $(0<\alpha<1)$, if

$$\left\|\Delta_hD^rf(x)\right\|^2_{L^2(\mathbb{R}_+,W_2^r(D))}=O\left(h^{2\alpha}\right),\quad h\to0.$$

The main result of this work is the following:

THEOREM 4. *Let $f(x)\in L^2\left(\mathbb{R}_+,W_2^r\left(D\right)\right)$. The following are equivalents:*

1.
$$\left\|\Delta_hD^rf(x)\right\|^2_{L^2(\mathbb{R}_+,W_2^r(D))}=O\left(h^{2\alpha}\right),\quad h\to0.$$

2.
$$\int_{u\geq N}\left|g(u)\right|^2du=O\left(N^{-2\alpha}\right),\quad N\to\infty.$$

PROOF. 1) *Necessity*: Suppose that for $f(x) \in L^2(\mathbb{R}_+, W_2^r(D))$ we have:

$$\|\Delta_h D^r f(x)\|^2_{L^2(\mathbb{R}_+, W_2^r(D))} = O\left(h^{2\alpha}\right), \quad h \to 0.$$

We know that for $hu \geq 0$

$$j_p(hu) = O(uh)^{-p-\frac{1}{2}},$$

hence:

$$\int_{u\geq N} |g(u)|^2\, du - \int_{u\geq N} j_p(hu)\, |g(u)|^2\, du = \int_{u\geq N} (1 - j_p(hu))\, |g(u)|^2\, du$$

$$= \int_{u\geq N} (1 - j_p(hu))\, |g(u)|^{2-a}\, |g(u)|^a\, du, \quad (\text{for } a \in \mathbb{R}^*)$$

$$\leq \left(\int_{u\geq N} |g(u)|^2\, du\right)^{\frac{2-a}{2}} \left(\int_{u\geq N} (1 - j_p(hu))^{\frac{2}{a}}\, |g(u)|^2\, du\right)^{\frac{a}{2}}, (\text{by Hölder inequality}),$$

$$= \left(\int_{u\geq N} |g(u)|^2\, du\right)^{\frac{2-a}{2}} \left(\int_{u\geq N} u^{4r+2p+1}(1 - j_p(hu))^{\frac{2}{a}}\, |g(u)|^2\, u^{-(4r+2p+1)} du\right)^{\frac{a}{2}}$$

$$\leq N^{\frac{a}{2}(-4r-2p-1)} \left(\int_{u\geq N} |g(u)|^2\, du\right)^{\frac{2-a}{2}} \left(\int_{u\geq N} u^{4r+2p+1}(1 - j_p(hu))^{\frac{2}{a}}\, |g(u)|^2\, du\right)^{\frac{a}{2}}.$$

Since $f \in W_2^r(D)$, then

$$\int_{u\geq N} u^{4r+2p+1}(1 - j_p(hu))^{\frac{2}{a}}\, |g(u)|^2\, du$$

$$\leq \int_0^\infty u^{4r+2p+1}(1 - j_p(hu))^{\frac{2}{a}}\, |g(u)|^2\, du.$$

For $a = 1$, we have

$$\int_{u\geq N} |g(u)|^2\, du - \int_{u\geq N} j_p(hu)\, |g(u)|^2\, du$$

$$\leq N^{\frac{1}{2}(-4r-2p-1)} \left(\int_{u\geq N} |g(u)|^2\, du\right)^{\frac{1}{2}} \left(\int_0^\infty u^{4r+2p+1}(1 - j_p(hu))^2\, |g(u)|^2\, du\right)^{\frac{1}{2}},$$

with

$$\int_0^\infty u^{4r+2p+1}(1 - j_p(hu))^2\, |g(u)|^2\, du = \|\Delta_h D^r f(x)\|^2,$$

then

$$\int_{u\geq N} |g(u)|^2\,du - \int_{u\geq N} j_p(hu)\,|g(u)|^2\,du$$
$$\leq N^{\frac{1}{2}(-4r-2p-1)}\left(\int_{u\geq N} |g(u)|^2\,du\right)^{\frac{1}{2}} \|\Delta_h D^r f(x)\|\,.$$

Hence

$$\int_{u\geq N} |g(u)|^2\,du \leq \int_{u\geq N} j_p(hu)\,|g(u)|^2\,du$$
$$+N^{\frac{1}{2}(-4r-2p-1)}\left(\int_{u\geq N} |g(u)|^2\,du\right)^{\frac{2-a}{2}} \|\Delta_h D^r f(x)\|^a\,,$$

and

$$\int_{u\geq N} |g(u)|^2\,du \leq O\,(Nh)^{-p-\frac{1}{2}}\int_{u\geq N} |g(u)|^2\,du$$
$$+N^{\frac{1}{2}(-4r-2p-1)}\left(\int_{u\geq N} |g(u)|^2\,du\right)^{\frac{1}{2}} \|\Delta_h D^r f(x)\|\,,$$

but

$$\left(1 - O\,(Nh)^{-p-\frac{1}{2}}\right)\int_{u\geq N} |g(u)|^2\,du$$
$$= O\left(N^{\frac{1}{2}(-4r-2p-1)}\right)\left(\int_{u\geq N} |g(u)|^2\,du\right)^{\frac{1}{2}} \|\Delta_h D^r f(x)\|\,.$$

We suppose

$$h = \frac{C}{N},\ \text{for all } C > 0, \quad \text{then } 1 - O\,(C)^{-p-\frac{1}{2}} \geq \frac{1}{2}.$$

So

$$\int_{u\geq N} |g(u)|^2\,du = O\left(N^{\frac{a}{2}(-4r-2p-1)}\right)\left(\int_{u\geq N} |g(u)|^2\,du\right)^{\frac{1}{2}} \|\Delta_h D^r f(x)\|\,,$$

then

$$1 = O\left(N^{\frac{1}{2}(-4r-2p-1)}\right)\left(\int_{u\geq N} |g(u)|^2\,du\right)^{-\frac{1}{2}} \|\Delta_h D^r f(x)\|\,,$$

which is equivalent to:

$$\left(\int_{u\geq N}|g(u)|^2\,du\right)^{\frac{1}{2}}=O\left(N^{\frac{1}{2}(-4r-2p-1)}\right)\|\Delta_h D^r f(x)\|\,,$$

and

$$\int_{u\geq N}|g(u)|^2\,du=O\left(N^{(-4r-2p-1)}\right)\|\Delta_h D^r f(x)\|^2\,.$$

For

$$\|\Delta_h D^r f(x)\|^2=O\left(h^{2\alpha}\right),$$

with $h=\frac{C}{N}, C>0$, C is chosen as following $C=N^{\frac{4r+2p+1}{\alpha}}>0$, and $N\geq 2$ such that

$$1-O\left(C\right)^{-p-\frac{1}{2}}\geq\frac{1}{2},$$

so

$$\begin{aligned}\int_{u\geq N}|g(u)|^2\,du &= O\left(N^{-(4r+2p+1)}N^{(4r+2p+1)}N^{-\alpha}\right)\\ &= O\left(N^{-\alpha}\right)\\ &= O\left(N^{-2\alpha}\right).\end{aligned}$$

2) *Sufficiency*: Suppose that

$$\int_{u\geq N}|g(u)|^2\,du=O\left(N^{-2\alpha}\right),\quad N\to\infty.$$

It is easy to show that it exists a function $f\in L^2\left(\mathbb{R}_+\right)$ such that $D^r f\in L^2\left(\mathbb{R}_+\right)$ and

$$D^r f(x)=\frac{1}{2^{2p}\Gamma(p+1)}\int_0^\infty j_p(th)j_p(tx)g(t)t^{2p+2r+1}dt.$$

Combining with the Parseval's identity, we obtain

$$\begin{aligned}\|\Delta_h D^r f(x)\|^2 &= \int_0^\infty |g(u)|^2(1-j_p(ux))^2u^{4r+2p+1}du\\ &= \int_{0<u<N}+\int_{u\geq N}\left[|g(u)|^2(1-j_p(ux))^2u^{4r+2p+1}du\right]\\ &= I_1+I_2,\end{aligned}$$

where $N = \left[\frac{1}{N}\right]$.

We have

$$\begin{aligned}
I_2 &= \int_{u\geq N} |g(u)|^2(1-j_p(ux))^2 u^{4r+2p+1}du \\
&= O\left\{\int_{u\geq N} |g(u)|^2 u^{4r+2p+1}du\right\} \\
&= O\left\{\sum_{n=N}^{\infty}\int_n^{n+1} |g(u)|^2 u^{4r+2p+1}du\right\} \\
&= O\left\{\sum_{n=N}^{\infty} n^{4r+2p+1}\int_n^{n+1} |g(u)|^2 du\right\} \\
&= O\left\{\sum_{n=N}^{\infty} n^{4r+2p+1}\left(\int_n^{\infty} |g(u)|^2 du - \int_{n+1}^{\infty} |g(u)|^2 du\right)\right\} \\
&= O\left\{\sum_{n=N}^{\infty} n^{4r+2p+1}\int_n^{\infty} |g(u)|^2 du - \sum_{n=N}^{\infty} n^{4r+2p+1}\int_{n+1}^{\infty} |g(u)|^2 du\right\} \\
&= O\left\{N^{4r+2p+1}\int_N^{\infty} |g(u)|^2 du + \sum_{n=N}^{\infty}\left[(n+1)^{4r+2p+1} - n^{4r+2p+1}\right]\int_n^{\infty} |g(u)|^2 du\right\} \\
&= O\left\{N^{4r+2p+1}\int_N^{\infty} |g(u)|^2 du + \sum_{n=N}^{\infty} n^{4r+2p}\int_n^{\infty} |g(u)|^2 du\right\} \\
&= O\left\{N^{4r+2p+1}N^{-2\alpha}\right\} + O\left\{N^{4r+2p}N^{-2\alpha}\right\}.
\end{aligned}$$

with $h = \frac{C}{N}, C > 0$, C is chosen as following $C = h^{1+\alpha} > 0$, such that

$$1 - O\left(C\right)^{-p-\frac{1}{2}} \geq \frac{1}{2},$$

then

$$I_2 = O\left(h^{4r+2p+1}h^{\alpha(4r+2p+1)}h^{-(4r+2p+1)}h^{-2\alpha}h^{-2\alpha^2}h^{2\alpha}\right) +$$

$$+O\left(h^{4r+2p}h^{\alpha(4r+2p)}h^{-(4r+2p)}h^{-2\alpha}h^{-2\alpha^2}h^{2\alpha}\right).$$

$$\begin{aligned} I_2 &= O\left(h^{\alpha(4r+2p+1)}h^{-2\alpha^2}\right) + O\left(h^{\alpha(4r+2p)}h^{-2\alpha^2}\right) \\ &= O\left(h^{\alpha(4r+2p)}h^{-2\alpha^2}\right)(h^\alpha + 1) \\ &= O(h^\alpha). \end{aligned}$$

Now for I_1, we have:

$$I_1 = \int_{0<u<N} |g(u)|^2(1 - j_p(ux))^2 u^{4r+2p+1} du.$$

we say that for $0 \leq u \leq 1$:

$$1 - j_p(ux) = O\left(h^2\right),$$

so

$$\begin{aligned} I_1 &= O\left(h^4\right) \int_{0<u<N} |g(u)|^2 u^{4r+2p+1} du \\ &= O\left(h^4\right) \sum_{n=0}^{N} \int_n^{n+1} |g(u)|^2 u^{4r+2p+1} du \\ &= O\left(h^4\right) \sum_{n=0}^{N} (n+1)^{4r+2p+1} \int_n^{n+1} |g(u)|^2 du \\ &= O\left(h^4\right) \sum_{n=0}^{N} (n+1)^{4r+2p+1} \left[\int_n^{\infty} |g(u)|^2 du - \int_{n+1}^{\infty} |g(u)|^2 du\right] \\ &= O\left(h^4\right) \left\{\sum_{n=0}^{N} (n+1)^{4r+2p+1} \int_n^{\infty} |g(u)|^2 du - \sum_{n=0}^{N} (n+1)^{4r+2p+1} \int_{n+1}^{\infty} |g(u)|^2 du\right\} \\ &= O\left(h^4\right) \left\{1 + \sum_{n=0}^{N} \left((n+1)^{4r+2p+1} - n^{4r+2p+1}\right) \int_n^{\infty} |g(u)|^2 du\right\} \\ &= O\left(h^4\right) \left\{1 + \sum_{n=0}^{N} n^{4r+2p} n^{-2\alpha}\right\} \\ &= O\left(h^4\right) \left(1 + N^{4r+2p} N^{-2\alpha}\right), \end{aligned}$$

with $N = \frac{C}{h}$, $C > 0$, C is chosen as following $C = h^{-1} > 0$, such that

$$1 - O(C)^{-p-\frac{1}{2}} \geq \frac{1}{2},$$

then

$$\begin{aligned} I_1 &= O\left(h^4\right)\left(1+h^{4\alpha}\right) \\ &= O\left(h^{2\alpha}\right). \end{aligned}$$

Hence

$$\left\|\Delta_h D^r f(x)\right\|^2 = O\left(h^{2\alpha}\right).$$

□

REMARK 1. If we consider the space $W_2^{r,k}(D)$, $k \geq 1$, we obtain the following result.

THEOREM 5. *For $k \geq 1$, if $f \in L^2\left(\mathbb{R}_+, W_2^{r,k}(D)\right)$ then the following are equivalents:*

1.

$$\left\|\Delta_h^k D^r f(x)\right\|^2 = O\left(h^{2k\alpha}\right), \quad 0 < \alpha < 1.$$

2.

$$\int_{u \geq N} |g(u)|^2\, du = O\left(N^{-2\alpha k}\right).$$

References

[1] Daher R. and Hamma M. An analog of Titchmarch's theorem of Jacobi transform. Int. J. Math. Anal. 2012; (6)17-20: 975-981.

[2] Ditkine V. Calcul opérationnel. UMN 2 1947; vip. 6 (22):72-158.

[3] Kontorovitch L. On approximate calculation of certain types of definite integrals and other applications of the method of selection of singularities. Mat. Sb. 1934; v. (41):235-245.

[4] James J. F. A Student's guide to Fourier transform with applications in Physics and Engineering. Third edition, Cambridge University Press, 2011.

[5] Kikushi H. Bessel transforms. Bull. Electrot Lab. 1954; (16), 111-120.

[6] Lébedev N. Certaines transformations intégrales de la physique mathématique. Thèse LGU, 1950.

[7] Maslouhi M. An analog of Titchmarsh's theorem for Dunkl transform. Integral transfomrs Spec. Functions 2010; (21) 9-10, 771-981.

[8] Nikol'skii S. M. Approximation of functions of several variables and imbedding and theorems. Springer-Verlag, New York, 1975.

[9] Platonov S. S.: An analog of Titchmarsh's theorem of the Fourier-walsh transform. Mathematical notes 2018; Vol. (103), N°1, 96-103.

[10] Titchmarsh E. C. Introduction to the theory of Fourier integrals. Oxford University Press, Amen House, London, E. C. 4, 1948.

[11] Zygmund A. Trigonometric series. Cambridge University Press, 1968.

Boudref Mohamed-Ahmed
Université de Bouira, Faculté des sciences et des sciences appliquées, Département des mathématiques, 10000, Algérie.
Laboratoire des Mathématiques appliquées, Faculté des Sciences Exactes, Université de Béjaia, 06000, Algérie
E-mail: m.boudref@univ-bouira.dz

Received
21.01.2019