Solving optimal control problems using the Picard's Iteration Method

Abderrahmane Akkouche[a, b]*, and Mohamed Aidene[b]
[a] Département de Mathématiques, Faculté des Sciences et des Sciences Appliquées, Université AKLI MOHAND OULHADJ de Bouira, 10 000 Bouira, Algérie.
[b] Laboratoire de Conception et Conduite des Systèmes de Production Université MOULOUD MAMMERI de Tizi-Ouzou, 15 000 Tizi-Ouzou, Algérie.

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Abstract

In this paper, the Picard's iteration method is proposed to obtain an approximate analytical solution for linear and nonlinear optimal control problems with quadratic objective functional. It consists in deriving the necessary optimality conditions using the minimum principle of Pontryagin, which result in a two-point-boundary-value-problem (TPBVP). By applying the Picard's iteration method to the resulting TPBVP, the optimal control law and the optimal trajectory are obtained in the form of a truncated series. The efficiency of the proposed technique for handling optimal control problems is illustrated by four numerical examples, and comparison with other methods is made. **keywords :**Optimal control, Pontryagin's minimum principle, Hamilton-Pontryagin equations, Picard's iteration method, Ordinary differential equations.

1 Introduction

Optimal control problems can be solved by direct or indirect methods [37]. Direct methods consist in converting the optimal control problem into an optimization one by discreetizing the state and the control variables, then the optimal control law is achieved using optimization methods [6]. However, the indirect methods consist in solving the necessary optimality conditions obtained from the application of the Pontryagin's minimum principle [31]. This necessary optimality conditions are given by a set of first order ordinary differential equations, known as the Hamilton-Pontryagin equations, with appropriate boundary conditions that define a two-point-boundary-value-problem (TPBVP) [4]. In which, the optimal control law is determined by solving the resulting TPBVP using the shooting methods [21, 39]. But the shooting methods suffer from difficulties in finding an approximate initial guess for the unspecified conditions at one end that produce solution reasonably close to the specified condition at the other end, because the solution is often very sensitive to small changes in the unspecified boundary conditions [6, 5]. These numerical difficulties are augmented by the relatively small domain of convergence of the Newton method which is built in the shooting methods [7].

In the last decade, a variety of semi-analytic methods to solve linear and nonlinear ordinary differential equations are presented [32]. These methods use practical iterative formulas to determine the solution or the approximate one of the problem in the form of an inifinite series that converges to the

*Author to whom the correspondence should be addressed : akkouche.abdo@yahoo.fr

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exact solution if it exists. The well known and utilised methods are the variational iteration method [15, 16], the homotopy perturbation method [14, 17, 18], the homotopy analysis method [27], and the differential transform method [20, 13].

These methods have been also used successfully to determine the solution of optimal control problems by solving the TPBVP obtained from the application of the Pontryagin's minimum principle. For instance; the variational iteration method [1, 28, 42], the homotopy perturbation method [10], the homotopy analysis method [43], and the differential transform method [30]. The principle of these methods for handling the optimal control problems, is that they transfer the resulting TPBVP into an initial value problem (IVP), and to turn round the difficulties of finding the initial guess, the unspecified initial conditions are selected as unknown parameters to be determined by imposing the boundary conditions. Thus, they constitute an interesting alternative to the shooting methods.

An other powerful technique, which has been applied successfully to solve both linear and non-linear ordinary differential equations is the Picard's iteration method (PIM) [33, 35]. Picard's iteration method used a simple iterative scheme to generate a sequence of approximations that converges to the exact solution provided that the resulting mapping is Lipschitz continuous and contractive [32]. The PIM method is demonstrated by many authors to be effective, and can easily handle a wide class of scientific and engineering applications [2, 3, 11, 25, 33, 34, 35, 36, 23]. In contrast to the most commonly used method, the proposed method can be applied directly for all type of differential equations, homogeneous, non homogeneous, linear and non linear as well. An other advantage, the method tackles the problem directly without any restrictive assumption such as small perturbation as the homotopy perturbation method, and does not requires the evaluation of the Lagrange multiplier like the variational iteration method.

The contribution of this paper is to use and assess the numerical performance of the Picard's iteration method for solving linear and nonlinear optimal control problems with quadratic objective functional that involve general boundary conditions. The Picard's iteration method is applied to achieve an approximate analytical solution of the Hamilton-Pontryagin equations iteratively given by a truncated power series. Since the Hamilton-Pontryagin equations are first order ordinary differential equations, the unspecified initial conditions are chosen as unknown parameters and determined by solving a set of algebraic equations. Moreover, the method allows to overcome the instability of the forward integration of the adjoint equation. All these advantages make this method an interesting tool for solving optimal control problems.

The rest of the paper is structured as follows. In Section 2, the optimal control problem considered in this study is formulated. Section 3 is devoted to the necessary optimality conditions of the formulated optimal control problem derived using the Pontryagin's minimum principle. The principle of the Picard's iteration method is explained in Section 4. A proposed design approach to solve optimal control problems using the Picard's iteration method is summarized in Section 5 and illustrated by four application examples in Section 6. Finally, Section 7 is reserved to the conclusion.

2 Statement of the problem

Consider the following optimal control problem with general boundary conditions :

$$\min_{u(t)} J(u(t)) = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t)\right) dt, \tag{1}$$

subject to

$$\dot{x}(t) = f(t,\, x(t),\, u(t)), \tag{2}$$

$$x(t_0) \in M_0, \tag{3}$$

$$x(t_f) \in M_1, \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and control vectors, respectively. $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector function which is continuously differentiable with respect to t, x and u. $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are positive semi-definite and positive definite matrices, respectively. t_0 and t_f denote the initial and final time, respectively, which assumed to be fixed. M_0 and M_1 are manifolds in $\mathbb{R}^n$ given as follows :

$$M_0 = \{x(t) \in \mathbb{R}^n | \Psi_1(x(t)) = \Psi_2(x(t)) = \cdots = \Psi_q(x(t)) = 0\},$$
$$M_1 = \{x(t) \in \mathbb{R}^n | \Phi_1(x(t)) = \Phi_2(x(t)) = \cdots = \Phi_l(x(t)) = 0\},$$

where the functions $\Psi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, q$ $(q \leq n)$ and $\Phi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, l$ $(l \leq n)$ are assumed to be continuously differentiable [41].

3 Necessary optimality conditions

To derive the necessary optimality conditions for an extremum of the optimal control problem (1)-(4), first we define the Hamiltonian function H as :

$$H(x(t),\, u(t),\, p(t),\, t) = \lambda_0 \left(\frac{1}{2} \left(x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t) \right) \right) + p^T(t)\, f(t,\, x(t),\, u(t)),$$

where λ_0 is a nonnegative constant, and $p(t) \in \mathbb{R}^n$ is the adjoint vector. λ_0 is equal to zero in the degenerate cases; otherwise $\lambda_0 \neq 0$ and can be normalised to a unit value, i.e., $\lambda_0 = 1$ [8, 9], and the above Hamiltonian function can be reduced to :

$$H(x(t),\, u(t),\, p(t),\, t) = \frac{1}{2} \left(x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t) \right) + p^T(t)\, f(t,\, x(t),\, u(t)). \tag{5}$$

In the future, whenever the multiplier λ_0 is not explicitly written, is assumed to be equal to 1.
According to the minimum principle of Pontryagin [31], the optimal control law is given by the minimisation of the Hamiltonian function (5) as follows :

$$\frac{\partial H}{\partial u}(x(t),\, u(t),\, p(t),\, t) = 0 \Rightarrow u(t) = \chi(x(t),\, p(t),\, t). \tag{6}$$

Substituting the expression of the optimal control law (6) into the Hamiltonian function (5), yields :

$$H^*(x(t),\, p(t),\, t) = H(x(t),\, \chi(x(t),\, p(t),\, t),\, p(t),\, t), \tag{7}$$

and the Hamilton-Pontryagin equations are given as :

$$\dot{x}(t) = \frac{\partial H^*}{\partial p(t)}(x(t),\, p(t),\, t), \tag{8}$$
$$\dot{p}(t) = -\frac{\partial H^*}{\partial x(t)}(x(t),\, p(t),\, t), \tag{9}$$

subject to the following boundary conditions :

$$x(t_0) \in M_0. \tag{10}$$
$$x(t_f) \in M_1, \tag{11}$$
$$p(t_0) = \sum_{i=1}^{q} \mu_i\, \nabla \Psi_i(x(t_0)), \tag{12}$$
$$p(t_f) = -\sum_{j=1}^{l} \eta_j\, \nabla \Phi_j(x(t_f)), \tag{13}$$

where $\mu = [\mu_1,\, \mu_2, \cdots, \mu_q]^T$ and $\eta = [\eta_1,\, \eta_2, \cdots, \eta_l]^T$ are the vectors of additional Lagrange multipliers associated with $\Psi = (\Psi_1,\, \Psi_2, \cdots, \Psi_q)$ and $\Phi = (\Phi_1,\, \Phi_2, \cdots, \Phi_l)$, respectively [24, 41].

Remark 1 *If M_0 is reduced to a single point, that is $M_0 = \{x_0\}$, where $x_0 = x(t_0)$, then the condition* (12) *is vacuous; if $M_0 = \mathbb{R}^n$, i.e., the initial point is not specified, we get $p(t_0) = 0$.*
Likewise, if $M_1 = \{x_1\}$ where $x_1 = x(t_f)$ then the condition (13) *is vacuous, and if $M_1 = \mathbb{R}^n$, i.e., the final point is free, then we have $p(t_f) = 0$. [26, 41]*

In this paper, it is proposed to use the Picard's iteration method to achieve an approximate analytical solution for the Hamilton-Pontryagin equations (8) – (9) that will be explained in the next section.

4 Picard's Iteration Method

To illustrate the basic idea of the Picard's iteration method [32, 33], consider the following initial value problem :

$$y'(t) = g(t, y(t)), \quad t > t_0, \tag{14}$$

$$y(t_0) = y_0, \tag{15}$$

where g is a continuous function. By integrating both sides of equation (14) with respect to t, yields the following integral equation :

$$y(t) = y_0 + \int_{t_0}^{t} g(\tau,\, y(\tau))d\tau. \tag{16}$$

To obtain the solution of the above integral equation (16) by means of the Picard's iteration method, we construct the following iteration formula :

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} g(\tau,\, y_k(\tau))d\tau, \tag{17}$$

and to start the process of resolution, we select a suitable initial approximation $y_0(t)$, which can be chosen as the initial condition of the problem, that is

$$y_0(t) = y_0, \quad \forall t \geq t_0,$$

and the other approximations $y_{k+1}(t), k \geq 0$ will be easily determined using the above iteration formula, and the solution of (14) is given as the limit of the sequence of functions $\{y_k\}$ generated by the formula (17), that is :

$$y(t) = \lim_{k \to \infty} y_k(t). \tag{18}$$

4.1 Convergence of PIM method for solving the Hamilton-Pontryagin equations

In order to solve the Hamilton-Pontryagin equations (8)-(9) subject to the appropriate boundary conditions by mean of the Picard's iteration method, we consider the following initial value problem :

$$\dot{x}(t) = \frac{\partial H^*}{\partial p(t)}(x(t),\, p(t),\, t), \tag{19}$$

$$\dot{p}(t) = -\frac{\partial H^*}{\partial x(t)}(x(t),\, p(t),\, t), \tag{20}$$

$$x(t_0) = x_0, \tag{21}$$

$$p(t_0) = p_0. \tag{22}$$

where x_0 and p_0 are the initial values of $x(t)$ and $p(t)$, respectively. We set

$$w(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}, \quad \Xi(t,\, w(t)) = \begin{pmatrix} \dfrac{\partial H^*}{\partial p(t)}(x(t),\, p(t),\, t) \\ \\ -\dfrac{\partial H^*}{\partial x(t)}(x(t),\, p(t),\, t) \end{pmatrix}, \tag{23}$$

and the system (19)–(22) is equivalent to :

$$\dot{w}(t) = \Xi(t,\, w(t)), \quad w(t_0) = w_0 = (x_0,\, p_0), \tag{24}$$

where $\Xi(t,\, w(t)) : [t_0,\, t_f] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is assumed to be continuous with its arguments. By applying the PIM method, the initial value problem (24) is equivalent to the following integral equation :

$$w(t) = w_0 + \int_{t_0}^{t} \Xi(\tau,\, w(\tau))\, ds, \tag{25}$$

which allows to generate a sequence of Picard iterates $w_k(t)$ on the interval $[t_0,\, t_f]$ by constructing the following iterative formula :

$$w_{k+1}(t) = w_0 + \int_{t_0}^{t} \Xi(\tau, w_k(\tau)) d\,\tau, \quad k \geq 0, \tag{26}$$

with $w_0(t) = w_0$, $\forall t \in [t_0,\ t_f]$. The relationship between the initial value problem (24) and the integral equation (25) is given by the following lemma [22].

Lemma 1 *[22]*
Assume that $\Xi : [t_0,\, t_f] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is continuous, and $(t_0,\, w_0) \in [t_0,\, t_f] \times \mathbb{R}^{2n}$, then $w(t)$ is a solution of the IVP (24) on the interval $[t_0,\, t_f]$ if and only if $w(t)$ is a solution of the integral equation (25) on the interval $[t_0,\, t_f]$.

Let $h = t_f - t_0$ and let us consider the complete normed space $C(I)$ of all real-valued continuous functions on the interval $I = [t_0,\, t_0 + h]$, with the norm $\parallel . \parallel$ defined by :

$$\parallel w(t) \parallel = \max_{t \in I} \mid w(t) \mid . \tag{27}$$

Based on the Picard-Lidelof theorem [22], we give the following sufficient condition for the convergence of the Picard's iteration method applied to the Hamilton-Pontryagin equations.

Theorem 1
Assume that $\Xi(t,\, w(t))$ is continuous $2n-$dimensional vector function on the parallelepiped

$$R \equiv \Big[(t,\, w(t)) :\ t_0 \leq t \leq t_0 + h,\ \parallel w(t) - w_0(t) \parallel \leq \beta\Big] \tag{28}$$

and suppose that :

- *$\Xi(t,\, w(t))$ satisfies a the following uniform Lipschitz condition with respect to $w(t)$ on R, i.e.,*

$$\parallel \Xi(t,\, w(t)) - \Xi(t,\, v(t)) \parallel \leq L \parallel w(t) - v(t) \parallel, \tag{29}$$

for all $(t,\, w(t)),\ (t,\, v(t)) \in R$, where $L > 0$ is the Lipschitz constant.

- $M = \max\Big[\parallel \Xi(t,\, w(t)) \parallel :\ (t,\, w(t)) \in R\Big]$.
- $h\,M \leq \beta$,
- $\mid h\,L \mid < 1$.

Then the sequence of Picard iterates $\{w_k(t)\}_{k=0}^{\infty}$ *given by*

$$w_{k+1}(t) = w_0 + \int_{t_0}^{t} \Xi(\tau, w_k(\tau))d\,\tau, \quad k \geq 0, \tag{30}$$

converges to the exact solution of the problem (24)

Corollary 1 *[22]*
Let $\{w_k\}$ *be the sequence of the Picard iterates generated by the iterative formula (26), and if* $w(t)$ *is the exact solution of (24), then*

$$\parallel w(t) - w_k(t) \parallel \leq \frac{M}{L}\,\frac{(L\,(t-t_0))^{k+1}}{(k+1)!} \tag{31}$$

for $t \in [t_0,\, t_0 + h]$*, where* L *is the Lipschitz constant for* $\Xi(t,\, w(t))$ *with respect to* w *on* R.

Theorem 2
Let $\{x_k(t)\}$ *and* $\{p_k(t)\}$ *be the solution sequences generated by the Picard's iteration formula (26) which converge respectively to* $x(t,\,\alpha)$ *and* $p(t,\,\alpha)$ *solution of the Hamilton-Pontryagin equation (8)-(9), as* $k \to \infty$*, where* α *is the vector of unknown parameters given either by* $x(t_0)$ *or* $p(t_0)$ *unspecified, which will be determined by imposing the boundary conditions (10)-(13). Then the sequences* $\{u_k(t)\}$ *and* $\{J(u_k(t))\}$ *defined by :*

$$u_k(t) = \chi(x_k(t),\, p_k(t),\, t), \tag{32}$$

$$J(u_k(t)) = \frac{1}{2}\int_{t_0}^{t_f} \Big(x_k^T(t)Qx_k(t) + u_k^T(t)Ru_k(t)\Big)\,dt, \tag{33}$$

converge to the optimal control law and optimal objective value, respectively.

Proof[38].

5 Proposed design approach

In this section a proposed design approach based on the PIM method to approximate the solution for the Hamilton-Pontryagin equations $(8) - (9)$ is proposed. The different steps of the proposed design approach can be summarized as follows.

Step 1 – Choose a desired threshold $\epsilon > 0$ and set $k = 0$.
Set the initial approximation $x_0(t) = x(t_0), \forall t \in [t_0,\, t_f]$ and $p_0(t) = p(t_0), \forall t \in [t_0,\, t_f]$. If the initial conditions are not specified, set $x_0(t) = \Lambda$ and $p_0(t) = \Theta$, where Λ and Θ are vectors of unknown parameters to be determined by imposing the boundary conditions (10)-(13).

Step 2 – Determine the approximate solution $x_{k+1}(t)$ and $p_{k+1}(t)$ using the iterative formula (26).

Step 3 – Impose the boundary conditions to determine the vectors of unknown parameters.

Step 4 – Deduce the optimal control $u_k(t)$ using the expression (6), and evaluate the performance index $J(u_k(t))$.

Step 5 – Stopping criterion. If

$$\frac{\mid J(u_{k+1}(t)) - J(u_k(t)) \mid}{\mid J(u_{k+1}(t)) \mid} \leq \epsilon, \tag{34}$$

stop, else set $k = k + 1$ and go to step (2).

6 Numerical Examples

In this section, the proposed approach to solve optimal control problems using the Picard's iteration method is illustrated by four application examples.

6.1 Example 1

Consider the following optimal control problem with unspecified initial conditions [12]

$$\min_{u(t)} J(u(t)) = \int_0^1 \left(x_1^2(t) + x_2^2(t) + u^2(t) \right) dt, \tag{35}$$

subject to

$$\dot{x}_1(t) = x_2(t) + u(t), \tag{36}$$
$$\dot{x}_2(t) = u(t), \tag{37}$$
$$x(0) \in M_0 \tag{38}$$
$$x_1(1) = free,\ x_2(1) = 1, \tag{39}$$

where M_0 is given as :

$$M_0 = \left\{ x(t) \in \mathbb{R}^2 | x_1(t) + x_2(t) - 3 = 0 \right\}.$$

The corresponding necessary optimality conditions are given as

$$\dot{x}_1(t) = x_2(t) - \frac{1}{2}\, p_1(t) - \frac{1}{2}\, p_2(t), \tag{40}$$
$$\dot{x}_2(t) = -\frac{1}{2}\, p_1(t) - \frac{1}{2}\, p_2(t), \tag{41}$$
$$\dot{p}_1(t) = -2\, x_1(t), \tag{42}$$
$$\dot{p}_2(t) = -2\, x_2(t) - p_1(t), \tag{43}$$

the optimal control law is given as :

$$u(t) = -\frac{1}{2}\, p_1(t) - \frac{1}{2}\, p_2(t). \tag{44}$$

The boundary conditions at $t = 0$ are :

$$x_1(0) + x_2(0) = 3, \tag{45}$$
$$p_1(0) = p_2(0) = \mu, \tag{46}$$

and the boundary condition at $t = 1$ are

$$x_2(1) = 1, \tag{47}$$
$$p_1(1) = 0. \tag{48}$$

Table 1: Obtained results for example 1

k	a	b	μ	J^{k+1}	$\dfrac{\mid J^{k+1} - J^k \mid}{\mid J^{k+1} \mid}$
0	0.66666667	2.3333333	1.3333333	5.740740667	–
1	1.7272727	1.2727273	2.3636364	4.402892551	0.3038566353
2	1.0909091	1.9090907	2.3496504	4.377023576	0.0059101749
3	1.2448980	1.7551020	2.4163265	4.168768892	0.0499559197
4	1.2038277	1.7961723	2.4186827	4.216459644	0.0113106150
5	1.2108197	1.7891803	2.4217869	4.206231839	0.0024315837
6	1.2095715	1.7904285	2.4218565	4.207906773	0.0003980444
7	1.2097244	1.7902756	2.4219336	4.207651508	0.0000606668
8	1.2097032	1.7902968	2.4219346	4.207681291	0.0000070782
9	1.2097052	1.7902948	2.4219357	4.207677665	0.0000008617

To determine an approximate solution of the Hamilton-Pontryagin equations $(40)-(43)$ with the PIM method, we construct the following iteration formulas :

$$x_1^{(k+1)}(t) = x_{10} + \int_0^t \left(x_2^{(k)}(\tau) - \frac{1}{2}\, p_1^{(k)}(\tau) - \frac{1}{2} p_2^{(k)}(\tau) \right) d\tau, \tag{49}$$

$$x_2^{(k+1)}(t) = x_{20} - \int_0^t \left(\frac{1}{2} p_1^{(k)}(\tau) + \frac{1}{2} p_2^{(k)}(\tau) \right) d\tau, \tag{50}$$

$$p_1^{(k+1)}(t) = p_{10} - 2 \int_0^t x_1^{(k)}(\tau)\, d\tau, \tag{51}$$

$$p_2^{(k+1)}(t) = p_{20} - \int_0^t \left(2\, x_2^{(k)}(\tau) + p_1^{(k)}(\tau) \right) d\tau. \tag{52}$$

Since the initial conditions for the state variables are not specified, then we set $x_1^{(0)}(t) = x_{10} = a$, $x_2^{(0)}(t) = x_{20} = b$, $p_1^{(0)}(t) = p_{10} = \mu$, and $p_2^{(0)}(t) = p_{20} = \mu$, where a, b and μ are unknown parameters to be determined by imposing the following conditions :

$$\begin{aligned} &x_1(0) + x_2(0) = 3, \\ &x_2(1) = 1, \\ &p_1(1) = 0. \end{aligned}$$

The obtained results are reported in Table 1. By assuming a threshold $\epsilon = 10^{-6}$, the PIM method converges after 10 iterations, and yields the following approximate optimal control law :

$$\begin{aligned} u(t) = &-2.42193570 + 4.21096785\, t - 2.13164090\, t^2 + 1.105273483\, t^3 - 0.2543594960\, t^4 \\ &+ 0.07543594960\, t^5 - 0.01103607501\, t^6 + 0.002276386308\, t^7 - 0.0002427410740\, t^8 \\ &+ 0.3828720250e - 4\, t^9 - 0.3204548672e - 5\, t^{10}, \end{aligned} \tag{53}$$

and the following approximate trajectories :

$$
\begin{aligned}
x_1(t) &= 1.2097052 - 0.6316409\,t + 0.8945160750\,t^2 - 0.0087189917\,t^3 + 0.09868162908\,t^4 \\
&+ 0.00439177498\,t^5 + 0.004094008393\,t^6 + 0.000219511893\,t^7 + 0.8747552070e - 4\,t^8 \\
&+ 0.464524605e - 5\,t^9 + 0.1131597205e - 5\,t^{10}, \qquad (54)\\
x_2(t) &= 1.7902948 - 2.4219357\,t + 2.105483925\,t^2 - 0.7105469666\,t^3 + 0.2763183707\,t^4 \\
&- 0.05087189919\,t^5 + 0.01257265826\,t^6 - 0.001576582144\,t^7 + 0.0002845482885\,t^8 \\
&- 0.2697123045e - 4\,t^9 + 0.3828720251e - 5\,t^{10}. \qquad (55)
\end{aligned}
$$

In figures (1) $-$ (2) the approximate solution obtained from the proposed method and those obtained using the shooting method [39, 21] and the sequential gradient restoration algorithm [12], are plotted, which show that the results are very close. It's worth to mention that the PIM method tackles the problem directly without any discreetization, and the results are readily obtained using few iterations. Furthermore, the proposed method facilitates the task of finding the initial approximations for the unspecified conditions by starting with unknown parameters that are determined by imposing the boundary conditions.

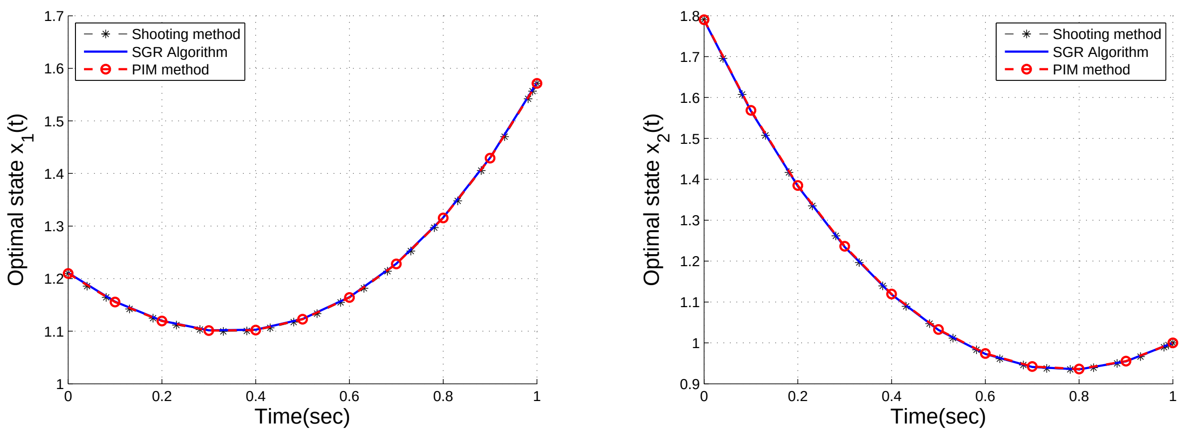

Figure 1: The state variables $x_1(t)$ and $x_2(t)$

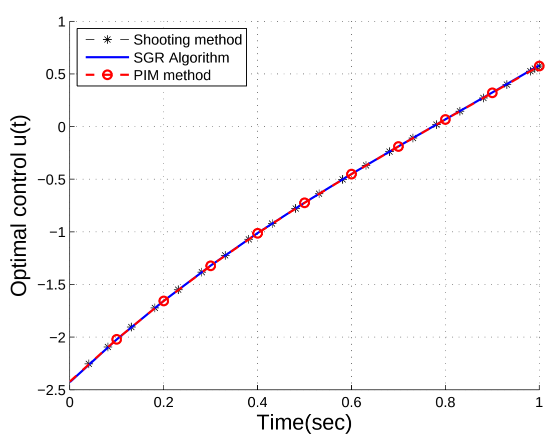

Figure 2: The control variable $u(t)$

6.2 Example 2

Consider the following optimal control problem with terminal constraint [24]

$$\min_{u(t)} J(u(t)) = \frac{1}{2}\int_0^2 u^2(t)\,dt, \tag{56}$$

subject to

$$\dot{x}_1(t) = x_2(t), \tag{57}$$
$$\dot{x}_2(t) = -x_2(t) + u(t), \tag{58}$$
$$x_1(0) = x_2(0) = 0, \tag{59}$$
$$x(2) \in M_1, \tag{60}$$

where M_1 is given as :

$$M_1 = \left\{x(t) \in \mathbb{R}^2 | x_1(t) + 5\,x_2(t) - 15 = 0\right\}.$$

The exact solution of this problem is given as :

$$x_1(t) = -1.379 + 0.894\,t + 1.136\,e^{-t} + 0.242\,e^t, \tag{61}$$
$$x_2(t) = 0.894 - 1.136\,e^{-t} + 0.242\,e^t, \tag{62}$$
$$u(t) = 0.894 + 0.484\,e^t. \tag{63}$$

The necessary optimality conditions are given by :

$$\dot{x}_1(t) = x_2(t), \tag{64}$$
$$\dot{x}_2(t) = -x_2(t) - p_2(t), \tag{65}$$
$$\dot{p}_1(t) = 0, \tag{66}$$
$$\dot{p}_2(t) = -p_1(t) + p_2(t), \tag{67}$$

subject to the following boundary conditions :

$$x_1(2) + 5\,x_2(2) = 15, \tag{68}$$
$$p_1(2) = -\eta, \tag{69}$$
$$p_2(2) = -5\,\eta. \tag{70}$$

To start the iterative process, we select $x_1^{(0)}(t) = x_1(0) = 0$, $x_2^{(0)}(t) = x_2(0) = 0$, $\forall t \in [0,\,2]$, and since $p_1(0)$ and $p_2(0)$ are unknown, we set $p_1^{(0)}(t) = p_1(0) = a$, $p_2^{(0)}(t) = p_2(0) = b$, $\forall t \in [0,\,2]$. Note that the initial approximations of the state variables don't satisfy the equation describing the manifold M_1 at $t = t_f = 2$, that is $x_1^{(0)}(2) + 5\,x_2^{(0)}(2) = 0 \neq 15$. While the next approximations will satisfy it at $t_f = 2$ because the unknown parameters will be determined by imposing the final conditions (68)–(70). Using these initial approximations, the iteration formulas for (64) – (67) are given as :

$$x_1^{(k+1)}(t) = \int_0^t \left(x_2^{(k)}(\tau)\right)\,d\tau, \tag{71}$$
$$x_2^{(k+1)}(t) = -\int_0^t \left(x_2^{(k)}(\tau) + p_2^{(k)}(\tau)\right)\,d\tau, \tag{72}$$
$$p_1^{(k+1)}(t) = a, \tag{73}$$
$$p_2^{(k+1)}(t) = b + \int_0^t \left(-p_1^{(k)}(\tau) + p_2^{(k)}(\tau)\right)\,d\tau, \tag{74}$$

Table 2: Obtained results for example 2

k	a	b	η	J^{k+1}	$\dfrac{\mid J^{k+} - J^{k} \mid}{\mid J^{k+1} \mid}$
0	−0.6428571429	−1.500000000	0.6428571429	5.801020410	−
1	−1.293103448	−2.327586207	1.293103448	17.86712248	0.6753243049
2	−0.7844036697	−1.279816514	0.7844036697	5.730832545	2.117718472
3	−0.9545454545	−1.500000000	0.9545454545	7.954420835	0.2795411930
4	−0.8767069422	−1.359297919	0.8767069422	6.532297350	0.2177064835
5	−0.9001015220	−1.389582712	0.9001015220	6.820202030	0.0422135119
6	−0.8930509388	−1.377026931	0.8930509388	6.694307340	0.0188062309
7	−0.8947388251	−1.379212791	0.8947388251	6.714490000	0.0030058366
8	−0.8943492284	−1.378519766	0.8943492284	6.707450615	0.0010494874
9	−0.8944243153	−1.378617010	0.8944243153	6.708329585	0.0001310266
10	−0.8944101529	−1.378591819	0.8944101529	6.708070840	0.0000385722
11	−0.8944124283	−1.378594766	0.8944124283	6.708097060	0.0000039087
12	−0.8944120652	−1.378594120	0.8944120652	6.708090370	0.0000009973

where a, b and η are obtained by imposing the boundary conditions $(68) - (70)$. Thus, the first iteration for $k = 0$ gives

$$\begin{aligned}
x_1^{(1)}(t) &= 0,\\
x_2^{(1)}(t) &= -b\,t,\\
p_1^{(1)}(t) &= a,\\
p_2^{(1)}(t) &= b - a\,t + b\,t,
\end{aligned}$$

with $a = -0.6428571429$, $b = -1.5$, hence the first approximate states are :

$$\begin{aligned}
x_1^{(1)}(t) &= 0,\\
x_2^{(1)}(t) &= 1.5\,t,
\end{aligned}$$

which yields

$$x_1^{(1)}(2) + 5\,x_2^{(1)}(2) = 15.$$

The results of the iteration process are reported in Table 2. By assuming a threshold $\epsilon = 10^{-6}$, the PIM method converges after 13 iterations. The obtained PIM solution and the analytical solution are plotted in figures $(3) - (4)$.

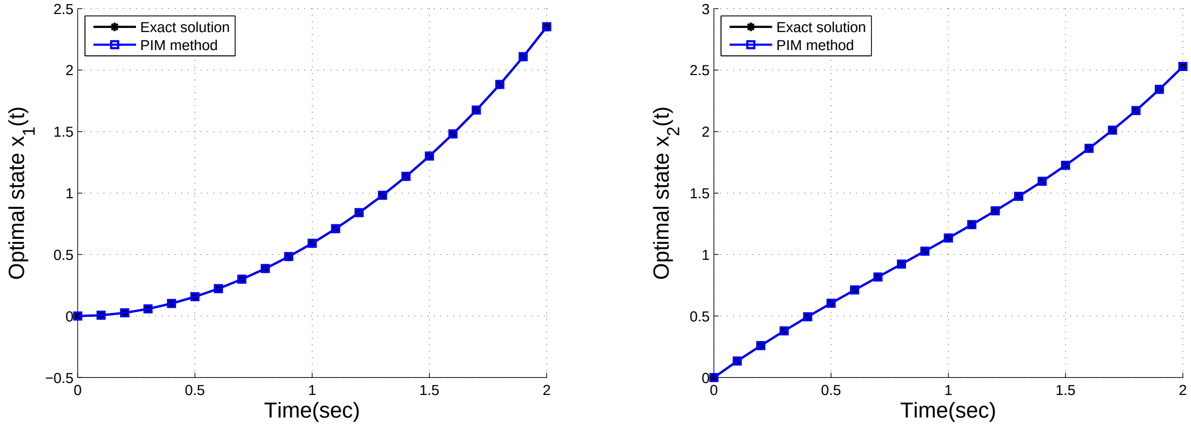

Figure 3: Exact and approximate solution for $x_1(t)$ and $x_2(t)$

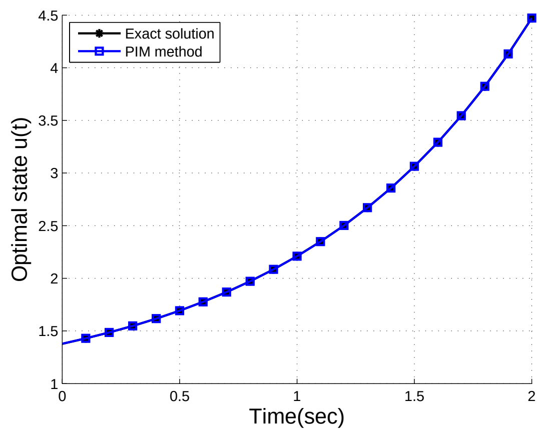

Figure 4: Exact and approximate solution for $u(t)$

6.3 Example 3

Consider the following optimal control problem of a vertical oven with three heating zones [40]

$$\min_{u(t)} = \frac{1}{2}\int_0^{10}\left[(x_1(t)-z_{1d})^2+(x_3(t)-z_{2d})^2+(x_5(t)-z_{3d})^2+10\,\frac{\parallel z_d \parallel^2}{\parallel u_d \parallel^2}\parallel u(t)-u_d \parallel^2\right]dt, \tag{75}$$

subject to

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{76}$$

with initial condition $x_1(0) = x_2(0) = \cdots = x_6(0) = 0$ and the final condition $x_1(10) = x_2(10) = \cdots = x_6(10) = 30$, where $z_d = (30°C,\, 30°C,\, 30°C)$ is the desired temperature, $u_d = (164.55,\, 245.30,\, 419.69)$ is the control which leads asymptotically to the prescribed temperature z_d.

The optimality conditions are given as :

$$\dot{x}(t) = A\,x(t) + B\,u_d - \frac{1}{l}\,B\,B^T\,p(t), \tag{77}$$

$$\dot{p}(t) = -A^T\,p(t) - C^T\,C\,x(t) + C^T\,z_d, \tag{78}$$

and the optimal control is :

$$u(t) = u_d - \frac{1}{l} B^T p(t). \tag{79}$$

where $l = 10 \dfrac{\| z_d \|^2}{\| u_d \|^2}$, $\quad A = \begin{pmatrix} -0.03 & 0.013 & 0.0077 & 0.0071 & 0.00017 & 0.00065 \\ 0.0017 & -0.012 & 0.00009 & 0.00033 & 0.00008 & 0.00029 \\ 0.0075 & 0 & -0.040 & 0.016 & 0.0077 & 0.00073 \\ 0 & 0.0030 & 0.0019 & -0.014 & 0.00009 & 0.0033 \\ 0 & 0 & 0.0075 & 0 & -0.029 & 0.012 \\ 0 & 0 & 0 & 0.0030 & 0.0014 & -0.013 \end{pmatrix}$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0.00125 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.00125 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.00125 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Starting from $x^{(0)}(t) = x(0) = (0, 0, 0, 0, 0, 0)^T$ and $p^{(0)}(t) = p(0) = \Lambda = (a, b, c, d, e, f)^T$, where the vector of unknown parameters Λ will be determined by imposing the final condition $x(10) = (30, 30, 30, 30, 30, 30)^T$. By assuming a threshold $\epsilon = 10^{-6}$, the method converges after 8 iterations. Simulation curves of the optimal control law $u_i(t)$, $i = 1, 2, 3$ and $x_i(t)$, $i = 1, \cdots, 6$, for $k = 8$ are shown in Figures (5) $-$ (6), respectively. Also, a comparison between the PIM solution with those obtained by the shooting method is made, which show that the results are very close to each other.

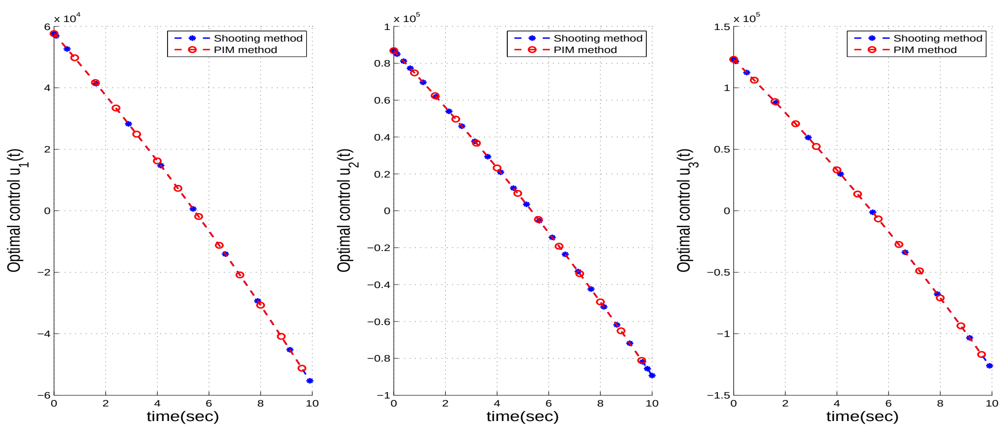

Figure 5: Graph of approximate control variables

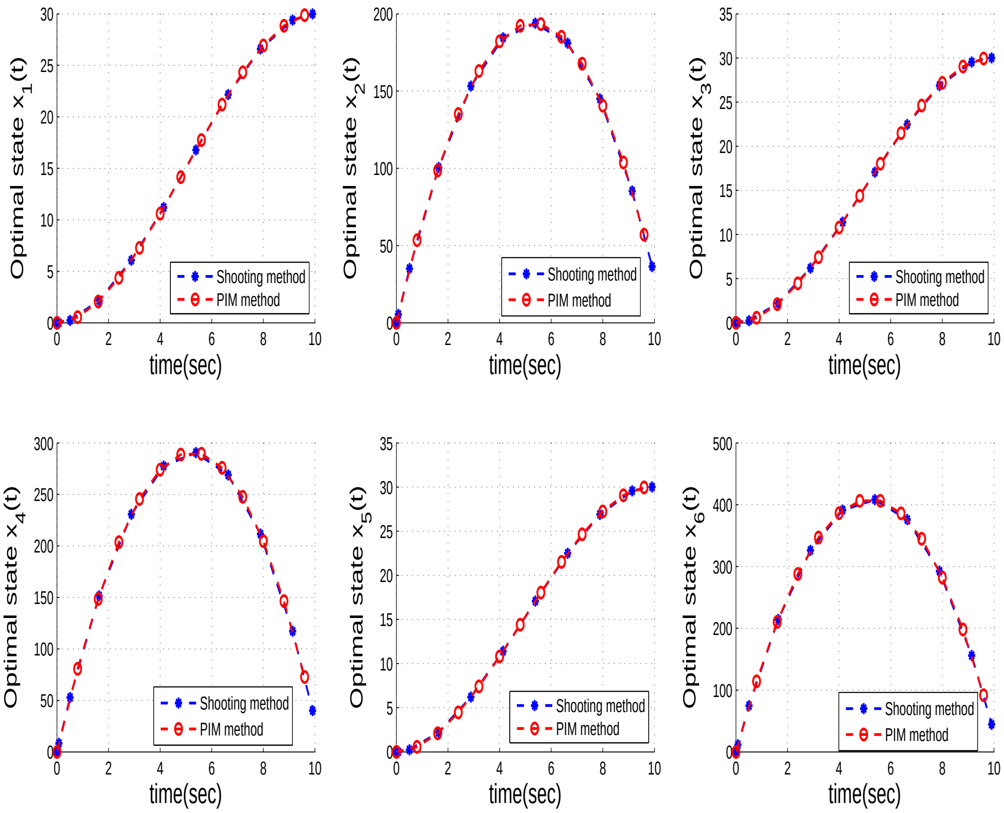

Figure 6: Graph of approximate state variables.

6.4 Example 4

Consider the following optimal maneuvers of a rigid asymmetric spacecraft [19].

$$\min_{u(t)} J(u(t)) = \int_0^{100} \frac{1}{2}\Big[u_1^2(t) + u_2^2(t) + u_3^2(t)\Big]\, dt \tag{80}$$

Subject to

$$\dot{x}_1(t) = -\frac{(I_3 - I_2)}{I_1} x_2(t)\, x_3(t) + \frac{1}{I_1}\, u_1(t), \tag{81}$$

$$\dot{x}_2(t) = -\frac{(I_1 - I_3)}{I_2} x_1(t)\, x_3(t) + \frac{1}{I_2}\, u_2(t), \tag{82}$$

$$\dot{x}_3(t) = -\frac{(I_2 - I_1)}{I_3} x_1(t)\, x_2(t) + \frac{1}{I_3}\, u_3(t), \tag{83}$$

with initial conditions $x(0) = (0.1, 0.005, 0.001)\, r/s$ and the final condition $x(100) = (0, 0, 0)\, r/s$. Where x_1, x_2 and x_3 are the angular velocities of spacecraft, u_1, u_2 and u_3 are the control torques. $I_1 = 86.24\, kgm^2$, $I_2 = 85.07\, kgm^2$, $I_3 = 113.59\, kgm^2$ are the spacecraft principle inertia.

By applying the minimum principle of Pontryagin, the necessary optimality conditions are given as :

$$\dot{x}_1(t) = -\frac{(I_3 - I_2)}{I_1}\, x_2(t)\, x_3(t) - \frac{1}{I_1^2}\, p_1(t), \tag{84}$$

$$\dot{x}_2(t) = -\frac{(I_1 - I_3)}{I_2}\, x_1(t)\, x_3(t) - \frac{1}{I_2^2}\, p_2(t), \tag{85}$$

$$\dot{x}_3(t) = -\frac{(I_2 - I_1)}{I_3}\, x_1(t)\, x_2(t) - \frac{1}{I_3^2}\, p_3(t), \tag{86}$$

$$\dot{p}_1(t) = \frac{(I_1 - I_3)}{I_2}\, x_3(t)\, p_2(t) + \frac{(I_2 - I_1)}{I_3} x_2(t)\, p_3(t), \tag{87}$$

$$\dot{p}_2(t) = \frac{(I_3 - I_2)}{I_1} x_3(t)\, p_1(t) + \frac{(I_2 - I_1)}{I_3} x_1(t)\, p_3(t), \tag{88}$$

$$\dot{p}_3(t) = \frac{(I_3 - I_2)}{I_1} x_2(t)\, p_1(t) + \frac{(I_1 - I_3)}{I_2} x_1(t)\, p_2(t), \tag{89}$$

with initial condition $x(0) = (0.1,\, 0.005,\, 0.005)^T$, and $p(0) = (a,\, b,\, c)^T$, where a,b and c are unknown parameters to be determined by imposing the final condition $x(100) = (0,\, 0,\, 0)$.
Also, the optimal control law is given as :

$$u_1(t) = -\frac{1}{I_1}\, p_1(t), \tag{90}$$

$$u_2(t) = -\frac{1}{I_2}\, p_2(t), \tag{91}$$

$$u_3(t) = -\frac{1}{I_3}\, p_3(t), \tag{92}$$

Based on the PIM method, the obtained results are reported in the Table 3.

Table 3: Iteration results for example 3.

k	a	b	c	J^{k+1}	$\frac{\mid J^{k+1} - J^k \mid}{\mid J^{k+1} \mid}$
0	0.7314359361	0.3851118899	0.1356718959	0.004689128998	−
1	0.7454116299	0.3581016008	0.1272376200	0.004685651750	$0.3477248e-5$
2	0.7438201686	0.3617129336	0.1281719579	0.004687107794	$0.1456044e-5$
3	0.7437324697	0.3618513022	0.1291057277	0.004687885574	$0.777780e-6$

By selecting a threshold $\epsilon = 10^{-6}$, the PIM method converges after 4 iterations, and the approximate solutions of trajectories $x_1(t)$, $x_2(t), x_3(t)$ and the control laws $u_1(t)$, $u_2(t)$ and $u_3(t)$, respectively, computed with the PIM method are plotted in Figures $(7) - (8)$.

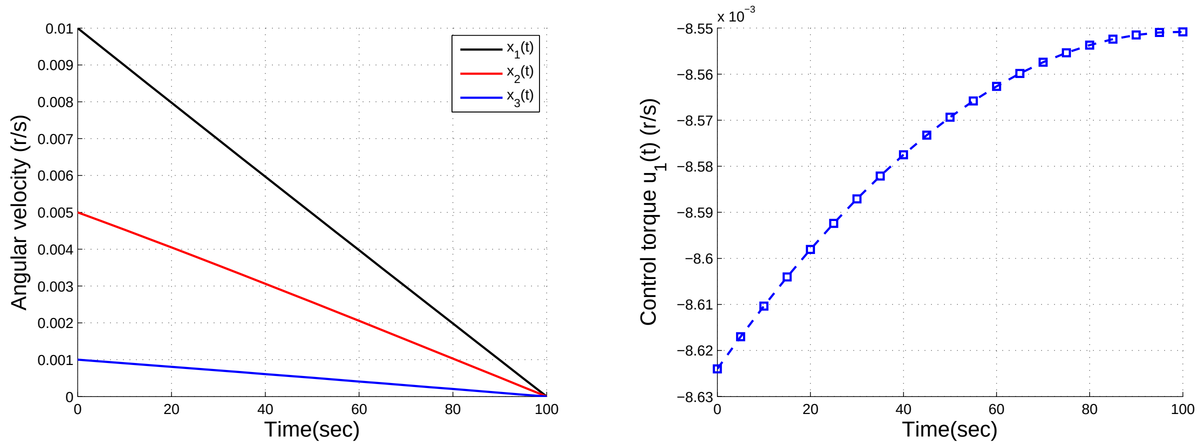

Figure 7: Graph of the trajectories and optimal control $u_1(t)$

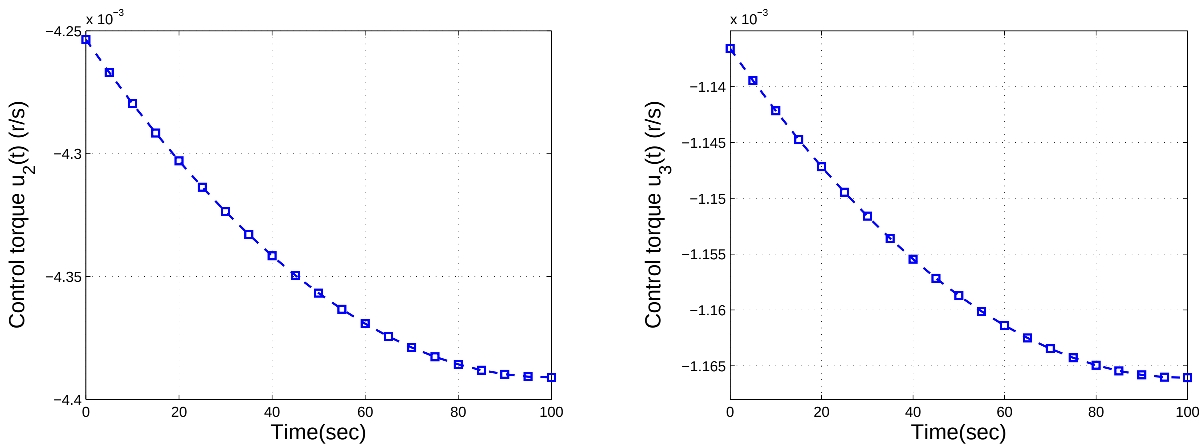

Figure 8: Graph of the optimal control $u_2(t)$ and $u_3(t)$

In Table 4, a comparison is made between the solution obtained by the present method with those obtained by Quasilinearisation and Chebyshev polynomials [19] and Composite Chebyshev finite difference method [29].

Table 4: Comparison results of the PIM method and Other Methods

The Method	Value of J
Quasilinearisation Chebyshev polynomials	0.004687
Composite Chebyshev finite difference method	0.004687
PIM method	0.004687

7 Conclusion

In this work, the Picard's iteration method is employed successfully to determine an approximate solution for a class of linear and nonlinear optimal control problems. The solution is obtained by solving iteratively the Hamilton-Pontryagin equations obtained from the Pontryagin's minimum principle. The solution of the Hamilton-Pontryagin equations is given in the form of a truncated series. The terms of the series are determined using practical iterative scheme starting by zeroth approximations. The zeroth approximations are chosen as the initial condition for the specified conditions, while for the unspecified conditions the zeroth approximations are chosen as unknown parameters, which are determined by solving a set of algebraic equations.

The PIM method provides an approximate solution with high degree of accuracy within few iterations, which means that the method converges rapidly. In addition, the method tackles the problem directly without any discreetization. Therefore, it is not affected by rounding the errors in the computational process. Also, the method attacks linear and nonlinear problems in the same manner unlike the Adomian Decomposition Method and the Homotopy perturbation method do, which need the calculation of polynomials which is a tedious task. The proposed method is illustrated by four numerical examples, and to demonstrate the effectiveness of the proposed method, a comparison is made between the obtained results and those obtained following other approaches which shows that are very close.

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