A primal-dual interior point algorithm for convex quadratic programming based on a new kernel function with an exponential barrier term

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Abstract

The introduction of kernel function in primal-dual interior point methods represents not only a measure of the distance between the iterate and the central path, but also plays an important role in the amelioration of the computational complexity of an interior point algorithm. In this paper, we present a polynomial primal-dual interior-point algorithm for solving convex quadratic programming based on a new kernel function with an exponential barrier term. It is shown that in the interior-point methods based on this function, the iteration bound enjoys $O(\sqrt{p^3 n}\,(\log pn)^2 \log \frac{n}{\epsilon})$ and $O(\sqrt{p^3 n} \log \frac{n}{\epsilon})$ for large and small-update methods respectively. This complexity generalizes the result obtained by Bai et al. and improves the results obtained by Bouafia et al..
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1 Introduction

Convex quadratic programs (CQP) is a generalization of linear optimization (LO). These programs appear in many areas of applications, for example in finance, agriculture, economics, optimal control, geometrics problems and also as sub-problems in sequential quadratic programming (SQP).

There are a variety of solution approaches for CQP which have been studied intensively. Among them, the interior point methods (*IPMs*) gained more attention than others methods. Feasible primal-dual path following methods are the most attractive methods of *IPMs* [11, 12]. Their derived algorithms achieved important results such as polynomial complexity and numerical efficiency. These algorithms trace approximately the so-called central-path which is a curve that lies in the feasible region of the considered problem and they

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reach an optimal solution of it. However, in practice these methods don't always find a strictly feasible centered point to starting their algorithms. So, it is worth analyzing other cases when the starting points are not centered. Thus leads to define a new method which is bases on finding initial strictly feasible points not necessarily centered.

These methods were studied extensively by many authors for linear optimization (LO) . Recently, Peng et al. [8] introduced so-called self-regular barrier functions for LO, the iteration bound for large-update methods was improved from $O(n\log\frac{n}{\varepsilon})$ to $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$, which almost closes the gap between the iteration bounds for large and small-update methods. Bai et al. [1] presented a large class of eligible kernel functions, which is fairly general and includes the classical logarithmic functions and the self-regular functions, as well as many non-self-regular functions as special cases. The best-known iteration bounds obtained are as good as the ones in [8] for appropriate choices of the eligible kernel functions. For some other related kernel function *IPMs* we refer to [2, 3, 4, 5, 6, 7, 10]. In 2016, Bouafia et al. [2] proposed a new parameterized kernel function with an exponential barrier that generalizes the algorithmic complexity obtained by Bai et al. [1], which has $O(\sqrt{p^5n}(\log pn)^2\log\frac{n}{\epsilon})$ complexity for large-update method and $O(\sqrt{p^5n}\log\frac{n}{\epsilon})$ for small-update method.

In this paper, we propose a primal-dual interior point method for solving CQP based on a new kernel function with an exponential barrier term, this function is used for determining the search directions and for measuring the distance between the given iterate and the center. We present some complexity results for the generic algorithm and prove that the bound for large and small-update methods enjoys $O(\sqrt{p^3n}\,(\log pn)^2\log\frac{n}{\epsilon})$ and $O(\sqrt{p^3n}\log\frac{n}{\epsilon})$, respectively. This complexity generalizes the result obtained by Bai et al. [1] and improved the results obtained by Bouafia et al. [2]

The primal problem of CQP is given by

$$\begin{cases} \min c^tx+\frac{1}{2}x^tQx, \\ Ax=b, \\ x\geq 0, \end{cases} \qquad (P)$$

where Q is a given $n\times n$ real positive semidefinite matrix, A is a given $m\times n$ real matrix, $c\in\mathbb{R}^n, b\in\mathbb{R}^m, x\in\mathbb{R}^n, z\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$.

The dual problem of (P) is given by

$$\begin{cases} \max b^ty-\frac{1}{2}x^tQx, \\ A^ty+z-Qx=c, \\ z\geq 0. \end{cases} \qquad (D)$$

Throughout the paper, we make the following assumptions:

1. The matrix A has full row rank $(rank(A)=m<n)$.
2. (P) and (D) satisfy the interior-point condition (IPC), i.e.,

there exist $\left(x^0, y^0, z^0\right)$ such that

$$\begin{cases} Ax^0 = b, \\ A^t y^0 + z^0 - Qx^0 = c, \\ x^0 > 0, z^0 > 0. \end{cases} \tag{1}$$

It is well known that finding an optimal solution for (P) and (D) is equivalent to solving the Karush-Khun-Tucker optimality conditions:

$$\begin{cases} Ax = b, x \geq 0, \\ A^t y + z - Qx = c, z \geq 0, \\ xz = 0. \end{cases} \tag{2}$$

Now, by replacing the complementarity condition $xz = 0$ in (2) by the perturbed equation $xz = \mu e$, one obtains the following perturbed system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^t y + z - Qx = c, z \geq 0, \\ xz = \mu e, \end{cases} \tag{3}$$

where μ is a positive parameter that is to be driven to zero explicitly. It is shown that, under our assumptions the system (3) has a unique solution, for each $\mu > 0$, this solution is denoted by $(x(\mu), y(\mu), z(\mu))$. $x(\mu)$ and $(y(\mu), z(\mu))$ are called the μ-center of (P) and (D), respectively. The set of all μ-centers forms the so called central path for (P) and (D).

If $\mu \longrightarrow 0$, then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for (P) and (D).

2 Primal-dual interior point algorithm for CQP

2.1 The search directions

Primal-dual path-following interior point algorithms are iterative methods which aim to trace approximately the central path, applying Newton's method for (3) for a given feasible point (x, y, z) then the Newton's direction $(\Delta x, \Delta y, \Delta z)$ at this point is the unique solution of the following linear system of equations:

$$\begin{cases} A\Delta x = 0, \\ A^t \Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \mu e - xz. \end{cases} \tag{4}$$

The result of a Newton step with step size α is denoted as

$$x^+ = x + \alpha\Delta x, \quad y^+ = y + \alpha\Delta y, \quad z^+ = z + \alpha\Delta z, \tag{5}$$

where α satisfies $0 < \alpha \leq 1$.

For convenience,we introduce the following notation. The vectors

$$v = \sqrt{\frac{xz}{\mu}}, d = \sqrt{\frac{x}{z}}.$$

The scaled search directions d_x and d_z as follows:

$$d_x = \frac{v\Delta x}{x}, d_z = \frac{v\Delta z}{z}. \tag{6}$$

System (4) can be rewritten as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^t\Delta y + d_z - \bar{Q}d_x = 0, \\ d_x + d_z = v^{-1} - v, \end{cases} \tag{7}$$

where $\bar{A} = \frac{1}{\sqrt{\mu}}AD, \bar{Q} = DQD$ with $D = diag(d)$.

Note that the right-hand side of the third equation in (7) equals the negative gradient of the logarithmic barrier function $\Phi(v)$, i.e., $d_x + d_z = -\nabla\Phi(v)$, and system (7) can be rewritten as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^t\Delta y + d_z - \bar{Q}d_x = 0, \\ d_x + d_z = -\nabla\Phi(v), \end{cases} \tag{8}$$

where the barrier function $\Phi(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is defined as follows:

$$\Phi(v) = \Phi(x, z; \mu) = \sum_{i=1}^{n} \psi(v_i), \tag{9}$$

$$\psi(v_i) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{10}$$

We use $\Phi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$, as follows:

$$\delta(v) = \frac{1}{2}\|\nabla\Phi(v)\| = \frac{1}{2}\|d_x + d_z\|. \tag{11}$$

We call $\psi(t)$ the kernel function of the logarithmic barrier function $\Phi(v)$. In this paper, we propose a new kernel function $\psi(t)$ non logarithmic and a new barrier function $\Phi(v)$, which will be defined in the next section.

2.2 The generic interior-point algorithm

It is clear from the above description that the closeness of (x, z) to $(x(\mu), z(\mu))$ is measured by the value of $\Phi(v)$ with $\tau > 0$ as a threshold value. If $\Phi(v) \leq \tau$, then we start a new outer iteration by performing a μ-update; otherwise, we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (5) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), z(\mu))$. Then μ is again reduced by the factor $1-\theta$ with $0 < \theta < 1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, say until $n\mu \leq \varepsilon$. At this stage, we have found an ε-approximate solution of CQP.

The parameters τ, θ and the step size α should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by algorithm is as small as possible.

The generic form of the algorithm is shown in Figure 1.

Generic primal-dual interior point algorithm for CQP

Input:
a proximity function $\Phi(v)$;
a threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$;
begin
$x = x^0, y = y^0, z = z^0, \mu = 1, v = \sqrt{\frac{xz}{\mu}}$.
$((x^0, y^0, z^0)$ is a strictly feasible solution for (P) and $(D))$
while $n\mu \geq \varepsilon$ do
begin (*outer iteration*)
$\mu = (1-\theta)\mu$;
while $\Phi(v) > \tau$ *do*
begin (*inner iteration*)
compute the direction $(\Delta x, \Delta y, \Delta z)$ *using* (8);
compute the step size α *and put*:
$x = x + \alpha\Delta x$;
$y = y + \alpha\Delta y$;
$z = z + \alpha\Delta z$;
$v = \sqrt{\frac{xz}{\mu}}$;
end
end
end.

Figure 1

3 New kernel function and its properties

3.1 New kernel function

In this section, we introduce a new parametric kernel function with an exponential barrier term and develop some useful properties of the new kernel function as well as the corresponding barrier function that are needed in the analysis of the algorithm.

We call $\psi : \mathbb{R}_{++} \to \mathbb{R}_{+}$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\begin{aligned} \psi(1) &= \psi^{'}(1) = 0, \\ \psi^{''}(t) &> 0, \forall t > 0, \\ \lim_{t\to 0^+} \psi(t) &= \lim_{t\to +\infty} \psi(t) = +\infty. \end{aligned}$$

Now, we define a new kernel function $\psi(t)$ as follows:

$$\psi(t) = \frac{p\left(t^2-1\right)}{2} - \int_1^t \frac{p}{x} e^{p\left(\frac{1}{x}-1\right)} dx = \int_1^t \left[p\left(x - \frac{1}{x} e^{p\left(\frac{1}{x}-1\right)}\right)\right] dx, \quad p \geq 2. \tag{12}$$

We give the first three derivatives with respect to t as follows:

$$\psi^{'}(t) = pt - \frac{p}{t} e^{p\left(\frac{1}{t}-1\right)}, \tag{13}$$

$$\psi^{''}(t) = p + e^{p\left(\frac{1}{t}-1\right)} \left(\frac{p}{t^2} + \frac{p^2}{t^3} \right), \tag{14}$$

$$\psi^{'''}(t) = -e^{p\left(\frac{1}{t}-1\right)} \left(\frac{2p}{t^3} + \frac{4p^2}{t^4} + \frac{p^3}{t^5} \right). \tag{15}$$

Obviously, we have:

$$\psi^{''}(t) > p > 0 \ , \forall t > 0, \tag{16}$$

$$\psi(1) = \psi^{'}(1) = 0. \tag{17}$$

It remains to show that $\psi(t)$ is a barrier function. On the one hand:

$$\lim_{t\to 0^+} \psi(t) = \lim_{t\to 0^+} \left[\frac{p\left(t^2-1\right)}{2} - \int_1^t \frac{p}{x} e^{p\left(\frac{1}{x}-1\right)} dx \right] = \frac{-p}{2} + \lim_{t\to 0^+} \left[\int_t^1 \frac{p}{x} e^{p\left(\frac{1}{x}-1\right)} dx \right].$$

Since $0 < x \leq 1$ and $p > 0$, we have:

$$\begin{aligned}
e^{p\left(\frac{1}{x}-1\right)} &\geq 1, \\
&\Longrightarrow \frac{p}{x}e^{p\left(\frac{1}{x}-1\right)} \geq \frac{p}{x}, \\
&\Longrightarrow \int_t^1 \frac{p}{x}e^{p\left(\frac{1}{x}-1\right)}dx \geq \int_t^1 \frac{p}{x}dx, \\
&\Longrightarrow \int_t^1 \frac{p}{x}e^{p\left(\frac{1}{x}-1\right)}dx \geq p\left[\log x\right]_t^1, \\
&\Longrightarrow \lim_{t\to 0^+}\psi(t) \geq \frac{-p}{2} + \lim_{t\to 0^+}(-p\log t) = +\infty,
\end{aligned}$$

so

$$\lim_{t\to 0^+}\psi(t) = +\infty. \tag{18}$$

On the other hand:

$$\lim_{t\to+\infty}\psi(t) = \lim_{t\to+\infty}\left[\frac{p\left(t^2-1\right)}{2} - \int_1^t \frac{p}{x}e^{p\left(\frac{1}{x}-1\right)}dx\right] = \lim_{t\to+\infty}\left[\int_1^t p\left[x - \frac{1}{x}e^{p\left(\frac{1}{x}-1\right)}\right]dx\right].$$

Since $1 \leq x < +\infty$ and $p > 0$, we have:

$$\begin{aligned}
e^{p\left(\frac{1}{x}-1\right)} &\leq 1, \\
&\Longrightarrow -\frac{1}{x}e^{p\left(\frac{1}{x}-1\right)} \geq -\frac{1}{x}, \\
&\Longrightarrow x - \frac{1}{x}e^{p\left(\frac{1}{x}-1\right)} \geq x - \frac{1}{x}, \\
&\Longrightarrow p\left(x - \frac{1}{x}e^{p\left(\frac{1}{x}-1\right)}\right) \geq p\left(x - \frac{1}{x}\right), \\
&\Longrightarrow \int_1^t p\left(x - \frac{1}{x}e^{p\left(\frac{1}{x}-1\right)}\right)dx \geq \int_1^t p\left(x - \frac{1}{x}\right)dx, \\
&\Longrightarrow \psi(t) \geq p\left[\frac{x^2}{2} - \log x\right]_1^t, \\
&\Longrightarrow \psi(t) \geq p\left(\frac{t^2}{2} - \log t - \frac{1}{2}\right), \\
&\Longrightarrow \lim_{t\to+\infty}\psi(t) \geq \lim_{t\to+\infty} pt\left(\frac{t}{2} - \frac{\log t}{t} - \frac{1}{2t}\right) = +\infty,
\end{aligned}$$

so

$$\lim_{t\to+\infty}\psi(t) = +\infty. \tag{19}$$

3.2 Eligibility of the new kernel function

Next lemma serves to prove the eligibility of our new kernel function (12).

Lemma 1 *Let* $\psi(t)$ *be as defined in* (12). *Then,*

$$\psi^{'''}(t) < 0, \forall t > 0, \tag{20}$$

$$t\psi^{''}(t) - \psi^{'}(t) > 0, \forall t > 0, \tag{21}$$

$$t\psi^{''}(t) + \psi^{'}(t) > 0, \forall t > 0, \tag{22}$$

$$\psi^{''}(t)\,\psi^{'}(\beta t) - \beta\psi^{'}(t)\,\psi^{''}(\beta t) > 0, \forall \beta > 1, \forall t > 1. \tag{23}$$

Proof. For (20) we use (15), we obtain $\psi^{'''}(t) < 0, \forall t > 0$. For (21) and (22), we use (13), (14) and the positivity of t, p and $e^{p\left(\frac{1}{t}-1\right)}$, we obtain:

$$\begin{aligned} t\psi^{''}(t) - \psi^{'}(t) &= t\left(p + e^{p\left(\frac{1}{t}-1\right)}\left(\frac{p}{t^2} + \frac{p^2}{t^3}\right)\right) - pt + \frac{p}{t}e^{p\left(\frac{1}{t}-1\right)}, \\ &= e^{p\left(\frac{1}{t}-1\right)}\left(\frac{2p}{t} + \frac{p^2}{t^2}\right) > 0. \end{aligned}$$

And

$$\begin{aligned} t\psi^{''}(t) + \psi^{'}(t) &= t\left(p + e^{p\left(\frac{1}{t}-1\right)}\left(\frac{p}{t^2} + \frac{p^2}{t^3}\right)\right) + pt - \frac{p}{t}e^{p\left(\frac{1}{t}-1\right)}, \\ &= 2tp + e^{p\left(\frac{1}{t}-1\right)}\left(\frac{p^2}{t^2}\right) > 0. \end{aligned}$$

For (23), ψ check(20) and (21). Let $t > 1$ be considered

$$f(\beta) = \psi^{''}(t)\,\psi^{'}(\beta t) - \beta\psi^{'}(t)\,\psi^{''}(\beta t), \forall \beta > 1,$$

we have:

$$f^{'}(\beta) = \psi^{''}(\beta t)\left[t\psi^{''}(t) - \psi^{'}(t)\right] - \beta t\psi^{'}(t)\,\psi^{'''}(\beta t), \forall \beta > 1,$$

therefore the function f is strictly increasing and $f(1) = 0$ then $f(\beta) > 0$.
This completes the proof. ∎

Lemma 2 [9] *Given a function* ψ *that is twice differentiable, then the following properties are equivalent*

(i) $\psi(\sqrt{t_1 t_2}) \leq \frac{\psi(t_1)+\psi(t_2)}{2}$.

(ii) *the function* ϕ *defined by* $\phi(\xi) = \psi(e^{\xi})$ *is convex.*

(iii) $t\psi^{''}(t) + \psi^{'}(t) > 0, t > 0$.

As a preparation for later, we present some technical results of the new kernel function.

Lemma 3 *For $\psi(t)$, we have*

$$\frac{p}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2p}\left[\psi'(t)\right]^2, t > 0. \tag{24}$$

$$\psi(t) \leq \frac{1}{2}\psi''(1)(t-1)^2 = \frac{p^2+2p}{2}(t-1)^2, t \geq 1. \tag{25}$$

Proof. For (24), using (16), we have:

$$\psi(t) = \int_1^t \int_1^x \psi''(y)\, dydx \geq \int_1^t \int_1^x p dydx = \frac{p}{2}(t-1)^2,$$

and

$$\begin{aligned}
\psi(t) &= \int_1^t \int_1^x \psi''(y)\, dydx, \\
&\leq \int_1^t \int_1^x \psi''(y) \frac{\psi''(x)}{p} dydx, \\
&\leq \frac{1}{p}\int_1^t \psi''(x)\psi'(x)\, dx = \frac{1}{2p}\left[\psi'(t)\right]^2.
\end{aligned}$$

For(25), since $\psi(1) = 0, \psi'(1) = 0, \psi'''(t) < 0, \psi''(1) = p^2 + 2p$, using Taylor's development we have for some $\zeta\,(1 \leq \zeta \leq t)$:

$$\begin{aligned}
\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\zeta)(\zeta-1)^3, \\
&\leq \frac{1}{2}\psi''(1)(t-1)^2 = \frac{p^2+2p}{2}(t-1)^2.
\end{aligned}$$

This completes the proof. ■

Let $\sigma : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$ and $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $\frac{-1}{2}\psi'(t)$ for all $t \in]0, 1]$, ρ is a decreasing function. Then we have the following lemma.

Lemma 4 *For $\psi(t)$, we have:*

$$1 + \sqrt{\frac{2}{p^2+2p}s} \leq \sigma(s) \leq 1 + \sqrt{\frac{2}{p}s}, s \geq 0. \tag{26}$$

$$\rho(z) \geq \frac{1}{\frac{2z}{p}+1}, z \geq 0. \tag{27}$$

Proof. For (26), let $\psi(t) = s$, $t \geq 1$, i.e., $t = \sigma(s), t \geq 1$. By (24), we have $\frac{p}{2}(t-1)^2 \leq \psi(t)$. Then $\frac{p}{2}(t-1)^2 \leq s$ this implies that $t = \sigma(s) \leq 1 + \sqrt{\frac{2}{p}s}$. By (25), we have:

$$s = \psi(t) \leq \frac{p^2 + 2p}{2}(t-1)^2, t \geq 1,$$

this implies

$$1 + \sqrt{\frac{2}{p^2 + 2p}s} \leq \sigma(s) = t.$$

For (27), let $z = \frac{-1}{2}\psi'(t)$, $t \in]0,1]$, i.e., $\rho(z) = t, t \in]0,1]$, we have: $e^{p\left(\frac{1}{t}-1\right)} \geq 1$. By the definition of $\psi'(t)$, we have:

$$\begin{aligned} z = \tfrac{-1}{2}\psi'(t) &= \tfrac{-1}{2}\left(pt - \tfrac{p}{t}e^{p\left(\frac{1}{t}-1\right)}\right), \\ &\geq \tfrac{-1}{2}pt + \tfrac{1}{2}\tfrac{p}{t} = \tfrac{p}{2}\left[(1-t) + \left(\tfrac{1}{t}-1\right)\right], \\ &\geq \tfrac{p}{2}\left(\tfrac{1}{t}-1\right). \end{aligned}$$

This implies that

$$t = \rho(z) \geq \frac{1}{\left(\frac{2z}{p}+1\right)}, z \geq 0.$$

This completes the proof. ■

Let $\psi_1(t) = \frac{p}{t}e^{p\left(\frac{1}{t}-1\right)}$, $p > 0$, $t \in]0,1]$ and let $\rho^* :]0,+\infty[\to]0,1]$ the inverse function of ψ_1, ρ^* is a decreasing function. Then we have the following lemma:

Lemma 5 *For $\psi_1(t)$, we have:*

$$\rho^*(z) \geq \frac{1}{\log\left(\frac{z}{p}\right)^{\frac{1}{p}} + 1}, z > 0. \tag{28}$$

$$\rho^*(p + 2z) \leq \rho(z), z \geq 0. \tag{29}$$

Proof. To show (28), let $\rho^*(z) = t$, i.e., $z = \psi_1(t) = \frac{p}{t}e^{p\left(\frac{1}{t}-1\right)}, p > 0$, for all $t \in]0,1]$. We have:

$$e^{p\left(\frac{1}{t}-1\right)} = \frac{tz}{p} \leq \frac{z}{p},$$

which implies

$$t = \rho^*(z) \geq \frac{1}{\log\left(\frac{z}{p}\right)^{\frac{1}{p}} + 1}, z > 0.$$

To show (29), let $\rho(z)=t$, i.e., $z=\frac{-1}{2}\psi^{'}(t)=\frac{-1}{2}(pt-\psi_1(t))$, for all $t\in]0,1]$. We have:

$$\begin{aligned}
z &= \frac{-1}{2}(pt-\psi_1(t)),\, z\geq 0,\\
z &\geq \frac{-1}{2}(p-\psi_1(t)),\, t\in]0,1],\\
p+2z &\geq \psi_1(t),\\
\rho^*(p+2z) &\leq t=\rho^*(\psi_1(t)) \ \ (\rho^* \text{ is decreasing}),\\
\rho^*(p+2z) &\leq \rho(z)\ .
\end{aligned}$$

This completes the proof. ■

We offer important theorem, which is valid for all kernel functions that satisfy (23)(Lemma 2.4 [1]).

Theorem 6 *[1] Let $\sigma:[0,+\infty[\rightarrow[1,+\infty[$ be the inverse function of $\psi(t)$ for $t\geq 1$. Then we have:*

$$\Phi(\beta v)\leq n\psi(\beta\sigma(\frac{\Phi(v)}{n})), v\in\mathbb{R}^n_{++}, \beta>1.$$

Corollary 7 *For any positive vector v, if $\Phi(v)\leq\tau$ and $\beta>1$, we have:*

$$\Phi(\beta v)\leq\frac{n}{2}\psi^{''}(1)\left(\beta\sigma\left(\frac{\tau}{n}\right)-1\right)^2.$$

Proof. For any positive vector v, If $\Phi(v)\leq\tau$ and $\beta>1$ then by the theorem 6, we have:

$$\Phi(\beta v)\leq n\psi(\beta\sigma(\frac{\Phi(v)}{n}))\leq n\psi(\beta\sigma(\frac{\tau}{n})).$$

ψ satisfies (25) and $\beta\sigma(\frac{\tau}{n})>1$ then $\psi(\beta\sigma(\frac{\tau}{n}))\leq\frac{\psi^{''}(1)}{2}\left(\beta\sigma(\frac{\tau}{n})-1\right)^2$. So $\Phi(\beta v)\leq\frac{n}{2}\psi^{''}(1)\left(\beta\sigma\left(\frac{\tau}{n}\right)-1\right)^2$.
This completes the proof. ■

Now let v^+ be the variance vector of (x,y,z) with respect to μ. Then one easily understands that the variance vector v^+ of (x,y,z) with respect to $\mu^+=(1-\theta)\mu$ is given by $v^+=\frac{v}{\sqrt{1-\theta}}$.

Lemma 8 *Let $0<\theta<1, v^+=\frac{v}{\sqrt{1-\theta}}$. If $\Phi(v)\leq\tau$, then we have:*

$$\Phi(v^+)\leq\frac{(p^2+2p)}{2(1-\theta)}(\theta\sqrt{n}+\sqrt{\frac{2}{p}\tau})^2.$$

Proof. By the corollary 7, with $\beta = \frac{1}{\sqrt{1-\theta}} > 1$ and (26), we obtain:

$$\begin{aligned}
\Phi(v^+) &\leq \frac{n}{2}\psi''(1)\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}-1\right)^2, \\
&= \frac{n\left(p^2+2p\right)}{2}\left(\frac{\sigma\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}-1\right)^2, \\
&= \frac{n\left(p^2+2p\right)}{2\left(1-\theta\right)}\left(\sigma\left(\frac{\tau}{n}\right)-\sqrt{1-\theta}\right)^2, \\
&\leq \frac{n\left(p^2+2p\right)}{2\left(1-\theta\right)}\left(1+\sqrt{\frac{2}{p}\left(\frac{\tau}{n}\right)}-\sqrt{1-\theta}\right)^2, \\
&\leq \frac{n\left(p^2+2p\right)}{2\left(1-\theta\right)}\left(\theta+\sqrt{\frac{2}{p}\left(\frac{\tau}{n}\right)}\right)^2, \ (1-\sqrt{1-\theta} \leq \theta, 0<\theta<1), \\
&\leq \frac{\left(p^2+2p\right)}{2\left(1-\theta\right)}\left(\theta\sqrt{n}+\sqrt{\frac{2}{p}\tau}\right)^2.
\end{aligned}$$

This completes the proof. ■

Denote

$$(\Phi)_0 = \frac{\left(p^2+2p\right)}{2\left(1-\theta\right)}\left(\theta\sqrt{n}+\sqrt{\frac{2}{p}\tau}\right)^2, \tag{30}$$

then $(\Phi)_0$ is an upper bound for $\Phi(v^+)$ during the process of the algorithm. Note that this bound depends only on the parameters n, τ and θ.

3.3 An estimation for the step size

In each inner iteration we first compute the search direction $\Delta x, \Delta y, \Delta z$ then a step size α. Recall that during an inner iteration the parameter μ is fixed. After a damped step, we have:

$$x^+ = x + \alpha\Delta x, \quad y^+ = y + \alpha\Delta y, \quad z^+ = z + \alpha\Delta z.$$

Using (6), we obtain:

$$\begin{aligned}
x^+ &= x\left(e+\alpha\frac{\Delta x}{x}\right) = x\left(e+\alpha\frac{d_x}{v}\right) = \frac{x}{v}\left(v+\alpha d_x\right), \\
z^+ &= z\left(e+\alpha\frac{\Delta z}{z}\right) = z\left(e+\alpha\frac{d_z}{v}\right) = \frac{z}{v}\left(v+\alpha d_z\right).
\end{aligned}$$

So, we have:

$$v_+ = \sqrt{\frac{x^+z^+}{\mu}} = \sqrt{\left(v+\alpha d_x\right)\left(v+\alpha d_z\right)}.$$

Define, for $\alpha > 0, f(\alpha) = \Phi(v_+) - \Phi(v)$. Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By (22) and lemma 2, we have:

$$\Phi(v_+) = \Phi(\sqrt{(v + \alpha d_x)(v + \alpha d_z)}) \leq \frac{1}{2}\left(\Phi\left(v + \alpha d_x\right) + \Phi\left(v + \alpha d_z\right)\right).$$

Therefore, we have:

$$f(\alpha) \leq f_1(\alpha) = \frac{1}{2}\left(\Phi\left(v + \alpha d_x\right) + \Phi\left(v + \alpha d_z\right)\right) - \Phi(v). \tag{31}$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to α, we have:

$$f_1^{'}(\alpha) \quad = \quad \frac{1}{2}\sum_{i=1}^{n}\Big(\psi^{'}(v_i + \alpha\,[d_x]_i)\,[d_x]_i + \psi^{'}(v_i + \alpha\,[d_z]_i)\,[d_z]_i\Big),$$

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\Big(\psi^{''}(v_i + \alpha\,[d_x]_i)\,[d_x]_i^2 + \psi^{''}(v_i + \alpha\,[d_z]_i)\,[d_z]_i^2\Big). \tag{32}$$

Using (8) and (11), we have

$$f_1^{'}(0) = \frac{1}{2}\left\langle \nabla\Phi(v), (d_x + d_z)\right\rangle = -\frac{1}{2}\left\langle \nabla\Phi(v), \nabla\Phi(v)\right\rangle = -2\left(\delta(v)\right)^2.$$

Lemma 9 *Let $\delta(v)$ be as defined in* (11)*. Then we have:*

$$\delta(v) \geq \sqrt{\frac{p}{2}\Phi(v)}. \tag{33}$$

Proof. Using (24), we have:

$$\Phi(v) = \sum_{i=1}^{n}\psi\left(v_i\right) \leq \sum_{i=1}^{n}\frac{1}{2p}\left[\psi^{'}\left(v_i\right)\right]^2 = \frac{1}{2p}\| \bigtriangledown \Phi(v)\|^2 = \frac{4}{2p}\delta(v)^2,$$

so

$$\delta(v) \geq \sqrt{\frac{p}{2}\Phi(v)}.$$

This completes the proof. ∎

Remark 10 *Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 9, the assumption that $\Phi(v) \geq \tau$ and $p \geq 2$, we have $\delta(v) \geq \sqrt{\frac{p}{2}} \geq 1$.*

For convenience, we denote $v_{\min} = \min_i(v_i), \delta = \delta(v)$ and $\Phi = \Phi(v)$.

Lemma 11 *Let $f_1(\alpha)$ be as defined in* (31) *and δ be as defined in* (11)*. Then we have:*

$$f_1^{''}(\alpha) \leq 2\delta^2\psi^{''}\left(v_{\min} - 2\alpha\delta\right).$$

Proof. According to the system (8), we observe that

$$(d_x)^t\, d_z = d_x^t \left(\bar{Q} d_x - \bar{A}^t \Delta y\right) = d_x^t \bar{Q} d_x \geq 0,$$

this implies that

$$\begin{aligned} 4\delta^2 &= \|d_x + d_z\|^2 = \|d_x\|^2 + \|d_z\|^2 + (d_x)^t\, d_z, \\ &\geq \|d_x\|^2 + \|d_z\|^2, \end{aligned}$$

so

$$\|d_x\| \leq 2\delta, \|d_z\| \leq 2\delta.$$

We have:

$$v_i + \alpha\, [d_x]_i \geq v_{\min} - 2\alpha\delta, v_i + \alpha\, [d_z]_i \geq v_{\min} - 2\alpha\delta, i = 1, .., n.$$

According to (20) ($\psi^{''}$ is strictly decreasing) and (32), we obtain:

$$\begin{aligned} f_1^{''}(\alpha) &\leq \frac{1}{2}\psi^{''}\left(v_{\min} - 2\alpha\delta\right) \sum_{i=1}^{n} \left([d_x]_i^2 + [d_z]_i^2\right), \\ &\leq \frac{1}{2}\psi^{''}\left(v_{\min} - 2\alpha\delta\right) \sum_{i=1}^{n} \left([d_x]_i + [d_z]_i\right)^2 \\ &= 2\delta^2 \psi^{''}\left(v_{\min} - 2\alpha\delta\right). \end{aligned}$$

This completes the proof. ■

Lemma 12 [*1*] *If* α *satisfies the inequality*

$$-\psi^{'}\left(v_{\min} - 2\alpha\delta\right) + \psi^{'}\left(v_{\min}\right) \leq 2\delta, \tag{34}$$

then

$$f_1^{'}(\alpha) \leq 0.$$

Lemma 13 [*1*] *The largest step size* $\bar{\alpha}$ *holding* (34) *is given by*

$$\bar{\alpha} = \frac{\left(\rho\left(\delta\right) - \rho\left(2\delta\right)\right)}{2\delta}.$$

Lemma 14 [*1*] *Let* $\bar{\alpha}$ *be as defined in Lemma 13. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)}.$$

Lemma 15 *Let* $\bar{\alpha}$ *be that defined in Lemma 13. If*

$$\Phi = \Phi(v) \geq \tau \geq 1,$$

so we have:

$$\bar{\alpha} \geq \frac{1}{p + (1+p)\,(p+4)\delta \left(1 + \log\left(\frac{p+4\delta}{p}\right)^{\frac{1}{p}}\right)^2}.$$

Proof. Using Lemma 14, we have $\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}$. According to (29) and the increase of the function $\frac{1}{\psi''}$, we obtain $\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{\psi''(\rho^*(p+2(2\delta)))}$.

So

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho^*(p+4\delta))}.$$

Let $t = \rho^*(p+4\delta)$ then we get $t \leq 1$ and

$$\begin{aligned}
\psi''(t) &= p + e^{p\left(\frac{1}{t}-1\right)}\left(\frac{p}{t^2} + \frac{p^2}{t^3}\right), \\
&= p + \frac{p}{t} e^{p\left(\frac{1}{t}-1\right)}\left(\frac{1}{t} + \frac{p}{t^2}\right), \\
&= p + \psi_1(t)\left(\frac{t+p}{t^2}\right), \\
&\leq p + \psi_1(t)\left(\frac{1+p}{t^2}\right), t \leq 1.
\end{aligned}$$

We have also according to (28) :

$$\begin{aligned}
\frac{1}{t^2} &= \frac{1}{(\rho^*(p+4\delta))^2}, \\
&\leq \left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2,
\end{aligned}$$

and

$$\psi_1(t) = \psi_1(\rho^*(p+4\delta)) = p + 4\delta.$$

Finally, we get

$$\begin{aligned}
\psi''(\rho^*(p+4\delta)) &\leq p + \psi_1(t)\left(\frac{1+p}{t^2}\right), \\
&\leq p + (1+p)(p+4\delta)\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2.
\end{aligned}$$

So we take

$$\begin{aligned}
\bar{\alpha} &\geq \frac{1}{\psi''(\rho^*(p+4\delta))}, \\
&\geq \frac{1}{p + (1+p)(p+4\delta)\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}, \\
&= \frac{1}{p + (1+p)\left(p\frac{1}{\delta} + 4\right)\delta\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}, \\
&\geq \frac{1}{p + (1+p)(p+4)\delta\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}, \delta \geq 1.
\end{aligned}$$

This completes the proof. $\blacksquare$

Denoting

$$\tilde{\alpha} = \frac{1}{p + (1+p)\,(p+4)\delta\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}. \tag{35}$$

$\tilde{\alpha}$ is the step of displacement and $\tilde{\alpha} \leq \bar{\alpha}$.

Lemma 16 [1] *If the step size α satisfies $\alpha \leq \bar{\alpha}$, then $f(\alpha) \leq -\alpha\delta^2$.*

Lemma 17 *For the displacement step, defined in* (35), *and taking*

$$\Phi(v) \geq 1.$$

So

$$f(\tilde{\alpha}) \leq \frac{-\sqrt{\frac{p}{2}\Phi}}{\left[\sqrt{2p} + (1+p)\,(p+4)\right]\left(1 + \log(\frac{p+4\sqrt{\frac{p}{2}(\Phi)_0}}{p})^{\frac{1}{p}}\right)^2}. \tag{36}$$

Proof. Using Lemma 16 with $\alpha = \tilde{\alpha}$ and (35). We obtain

$$\begin{aligned} f(\tilde{\alpha}) &\leq -\tilde{\alpha}\delta^2, \\ &= \frac{-\delta^2}{p + (1+p)\,(p+4)\delta\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}, \end{aligned}$$

because $\log(1 + \frac{4\delta}{p}) \geq 0$ and $\delta \geq \sqrt{\frac{p}{2}}$, we have

$$f(\tilde{\alpha}) \leq \frac{-\delta}{[p\sqrt{\frac{2}{p}} + (1+p)\,(p+4)]\left(1 + \log(\frac{p+4\delta}{p})^{\frac{1}{p}}\right)^2}.$$

Let the increasing function $g_1(x) = \frac{x}{\left(1+\log(\frac{p+4x}{p})^{\frac{1}{p}}\right)^2}, \forall x \in \mathbb{R}_{++}$ and we have $\delta \geq \sqrt{\frac{p}{2}\Phi}$ then

$$f(\tilde{\alpha}) \leq \frac{-\sqrt{\frac{p}{2}\Phi}}{[\sqrt{2p} + (1+p)\,(p+4)]\left(1 + \log(\frac{p+4\sqrt{\frac{p}{2}\Phi}}{p})^{\frac{1}{p}}\right)^2},$$

let the decreasing function $g_2(x) = \frac{1}{\left(1+\log(\frac{p+4x}{p})^{\frac{1}{p}}\right)^2}, \forall x \in \mathbb{R}_{++}$ and since $\Phi \leq (\Phi)_0$, we obtain

$$f(\tilde{\alpha}) \leq \frac{-\sqrt{\frac{p}{2}\Phi}}{\left[\sqrt{2p} + (1+p)\,(p+4)\right]\left(1 + \log(\frac{p+4\sqrt{\frac{p}{2}(\Phi)_0}}{p})^{\frac{1}{p}}\right)^2}.$$

This completes the proof. $\blacksquare$

4 Complexity of the algorithm

4.1 Inner iteration bound

After the update of μ to $(1-\theta)\mu$, we have:

$$\Phi\left(v^{+}\right) \leq \frac{\left(p^{2}+2p\right)}{2\left(1-\theta\right)}\left(\theta\sqrt{n}+\sqrt{\frac{2}{p}\tau}\right)^{2} = \left(\Phi\right)_{0}.$$

We need to count how many inner iterations are required to return to the situation where $\Phi \leq \tau$. We denote the value of $\Phi(v)$ after the μ update as $(\Phi)_0$; the subsequent values in the same outer iteration are denoted as $(\Phi)_k$, $k = 1, 2, .., K$, where K denotes the total number of inner iterations in the outer iteration. The decrease in each inner iteration is given by (36). In [1], we can find the appropriate values of $\bar{\kappa} > 0$ and $\gamma \in]0, 1]$ as:

$$\gamma = \frac{1}{2}, \bar{\kappa} = \frac{2\sqrt{p}}{\left[4\sqrt{p}+2\sqrt{2}\left(1+p\right)\left(p+4\right)\right]\left(1+\log\left(\frac{p+4\sqrt{\frac{p}{2}(\Phi)_0}}{p}\right)^{\frac{1}{p}}\right)^{2}}.$$

Lemma 18 *Let K be the total number of inner iterations in the outer iteration. Then we have:*

$$K \leq \frac{\left[4\sqrt{p}+2\sqrt{2}\left(1+p\right)\left(p+4\right)\right]}{\sqrt{p}}\left(1+\log\left(\frac{p+4\sqrt{\frac{p}{2}\left(\Phi\right)_0}}{p}\right)^{\frac{1}{p}}\right)^{2}\left(\Phi\right)_0^{1/2}.$$

Proof. By Lemma 1.3.2 in [1]

$$\begin{aligned} K &\leq \frac{\left((\Phi)_0\right)^{\gamma}}{\bar{\kappa}\gamma}, \\ &= \frac{\left[4\sqrt{p}+2\sqrt{2}(1+p)(p+4)\right]}{\sqrt{p}}\left(1+\log\left(\frac{p+4\sqrt{\frac{p}{2}(\Phi)_0}}{p}\right)^{\frac{1}{p}}\right)^{2}\left(\Phi\right)_0^{1/2}. \end{aligned}$$

This completes the proof. ■

4.2 Total iteration bound

The number of outer iterations is bounded above by $\frac{\log\frac{n}{\epsilon}}{\theta}$ (see [11] Lemma II.17). Through multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely:

$$\frac{\left[4\sqrt{p}+2\sqrt{2}\left(1+p\right)\left(p+4\right)\right]}{\sqrt{p}}\left(1+\log\left(\frac{p+4\sqrt{\frac{p}{2}\left(\Phi\right)_0}}{p}\right)^{\frac{1}{p}}\right)^{2}\left(\left(\Phi\right)_0\right)^{1/2}\frac{\log\frac{n}{\epsilon}}{\theta}. \quad (37)$$

For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have:
$O\left(\sqrt{p^3 n}\left(\log pn\right)^2\log\frac{n}{\epsilon}\right)$ iterations complexity.
For small-update methods with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have:
$O\left(\sqrt{p^3 n}\log\frac{n}{\epsilon}\right)$ iterations complexity.

5 Conclusions

In this paper we present a polynomial primal-dual interior-point algorithm for convex quadratic programming based on a new kernel function. This approach has the advantage of starting with any point (x^0, y^0, z^0) satisfying the condition IPC not necessary centred. The proposed kernel function has an exponential barrier term, witch is not logarithmic and not self-regular. We proved that the iteration bound of interior point method based on this function is $O(\sqrt{p^3 n}\,(\log pn)^2 \log \frac{n}{\epsilon})$ iterations complexity for large-update method and $O(\sqrt{p^3 n}\log \frac{n}{\epsilon})$ iterations complexity for small-update method. Our approach has generalized the result obtained by Bai et al. [1] and improved the results obtained by Bouafia et al. in [2].

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