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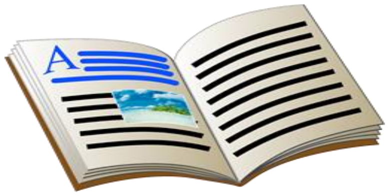
وزارة التعليم العالي والبحث العلمي
جامعة أكلي محند أولحاج
- البويرة -

Faculté des Sciences et des Sciences Appliquées

كلية العلوم والعلوم التطبيقية

Department of Physics

Course Handout : **Vibrations, Waves and Optics**



Speciality: Fundamental Chemistry

Level: 2nd year.

Vibrations, Waves and Optics

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Year: 2024/2025

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UEF12 / F124

Vibrations, Ondes and Optique

(1h30' Course+1h30' Tutorials / week) ; 45h00/Semester

Content of the subject:

1. Second-order differential equations with constant coefficients

1. 1. Homogeneous equation: Highly damped regime, Critical regime, Pseudoperiodic regime.

1.2 Equation with second member: General solution, special cases of a sinusoidal second member.

2. Free oscillations of one-degree-of-freedom systems

2.1 Undamped oscillations: Linear oscillator, differential equation of the simple harmonic oscillator, natural pulsation, energy.

2.2 Free oscillations of damped systems with one degree of freedom. Special case of viscous friction: Differential equation of motion, logarithmic decrement, quality coefficient.

3. Forced oscillations of one-degree-of-freedom systems

3.1 Differential equation of the mass-spring-damper system in forced oscillation:

3.2 Special case of the sinusoidal permanent regime. Mechanical impedance. Power. Resonance. Bandwidth. Quality coefficient.

4. Free oscillations of two-degree-of-freedom systems

4. 1 Mass-spring system in translation: Differential equations of motion. Concept of coupling. Proper pulsations. Proper modes. Beating phenomenon.

4.2. Coupled pendulums

5. General information on propagation phenomena

7.1 One-dimensional propagation: Propagation equation, Solution of the propagation equation, sinusoidal progressive wave, wavelength, wave number.

7.2 Linear chain model

6. Acoustic waves in fluids

8.1 Equation of propagation of acoustic waves in fluids, speed of sound.

8.2 Progressive sinusoidal wave: sound pressure, sound impedance, sound energy, sound intensity.

8.3 Reflection-Transmission of acoustic waves at normal incidence.

7. Principles and laws of geometric optics

9.1 Reflection – Refraction

9.2 Prism

8. Construction of images

10.1 Stigma

10.2 Plane and spherical diopters

10.3 Plane and spherical mirrors

10.4 Thin lenses.

Contents

Foreword

This course on Vibrations, Waves, and Optics is designed for second-year university students specializing in chemistry. The interplay of vibrational and wave phenomena is fundamental to a comprehensive understanding of both physical and chemical processes. As such, this course seeks to provide students with a rigorous foundation in the principles governing these phenomena, emphasizing their relevance to the field of chemistry.

Vibrations, as a manifestation of oscillatory motion, are intrinsic to molecular behaviour and play a crucial role in determining the properties of materials. The study of waves extends this understanding, illustrating how energy propagates through various media. In conjunction with optical principles, these topics offer significant insights into spectroscopic techniques, molecular interactions, and the nature of light-matter interactions.

The curriculum will cover essential theoretical frameworks, including harmonic motion, wave mechanics, and the principles of geometrical and physical optics. Students will engage with both theoretical and practical aspects of these topics through a blend of lectures, and problem-solving sessions. Additionally, these concepts will be applied in vibration spectroscopy during the third year. This comprehensive approach aims to cultivate critical thinking and analytical skills, which are vital for future research and professional pursuits in the field of chemistry.

Students are encouraged to actively participate in discussions and collaborate with peers to enhance their learning experience. The knowledge and skills acquired in this course will serve as foundational elements for advanced studies and research in various subfields of chemistry.

We anticipate that this course will not only deepen students' understanding of vibrations, waves, and optics but also stimulate their interest in the broader applications of these concepts within the discipline of chemistry.

Sincerely,

Dr BOUHDJER Lazhar
Departement of Physics University of Bouira 02/10/2024

General Introduction

The course *Vibrations, Ondes et Optique* is tailored for second-year chemistry students, offering a thorough exploration of the physical phenomena that play crucial roles in understanding molecular behavior, energy transfer, and chemical analysis techniques. The curriculum is structured to provide a deep understanding of key topics in vibrations, wave propagation, and optics, all of which are essential to many areas of chemistry.

The following chapters will be covered:

1. **Second-order Differential Equations with Constant Coefficients:**

   This chapter introduces second-order differential equations, which are fundamental in describing the behavior of oscillatory systems. A solid grasp of these mathematical tools is essential for analyzing mechanical and molecular vibrations.

2. **Free Oscillations of One-degree-of-freedom Systems:**

   We will explore free oscillations in systems with one degree of freedom, laying the foundation for understanding how molecules vibrate when undisturbed. This chapter will link oscillatory motion to energy levels in molecules and will be fundamental to topics such as infrared spectroscopy.

3. **Forced Oscillations of One-degree-of-freedom Systems:**

   Forced oscillations occur when an external force acts on a system. This chapter will analyze how systems respond to periodic driving forces, with applications in resonance phenomena, which are critical in spectroscopic techniques and molecular excitation.

4. **Free Oscillations of Two-degree-of-freedom Systems:**

   This chapter extends the concept of free oscillations to systems with two degrees of freedom, such as coupled oscillators, which serve as models for polyatomic molecular vibrations. Understanding these concepts is crucial for interpreting complex vibrational spectra.

5. **General Information on Propagation Phenomena:**

   Propagation of waves, whether mechanical or electromagnetic, is central to the transmission of energy. This chapter will cover the basic principles of wave propagation, including speed, wavelength, and frequency, which are vital in understanding both acoustic and optical phenomena.

6. **Acoustic Waves in Fluids:**

   This section focuses on the behavior of acoustic waves in fluids, examining how sound propagates through gases and liquids. Understanding acoustic waves has applications in chemical sensing and reaction monitoring.

7. **Principles and Laws of Geometric Optics:**

   Geometric optics is the study of light propagation in terms of rays. This chapter will cover key laws such as reflection and refraction, as well as the principles behind lenses and mirrors, which are essential for the design and understanding of optical instruments used in chemical analysis.

8. **Construction of Images:**

   This chapter deals with the formation of images using optical systems, focusing on how lenses and mirrors are used to manipulate light. The construction of images is critical to techniques such as microscopy, which is widely used in chemistry to study the structure of materials.

By the end of this course, students will have a firm grasp of the mathematical and physical principles underlying vibrations, wave propagation, and optics, as well as their applications in chemistry. The knowledge gained in these areas will be indispensable for further studies in molecular spectroscopy, reaction dynamics, and various chemical analysis techniques.

Chapter 1: Second-order differential equations with constant coefficients

I-1. Introduction:

Differential equations play a fundamental role in modeling a wide array of physical, chemical, and biological phenomena. In particular, second-order differential equations with constant coefficients are commonly encountered in the analysis of dynamic systems, including harmonic oscillators, electrical circuits, and mechanical systems.

This chapter will focus on the methods used to solve linear second-order differential equations with constant coefficients. It begins with a formal definition of these equations and an exploration of their various forms. Subsequently, we will discuss both general and particular solutions, which depend on the values of the coefficients and the initial conditions. These concepts will then be illustrated through practical examples and real-world applications.

The objective of this chapter is to provide a thorough understanding of the techniques for solving such equations and to demonstrate their applicability in modeling and analyzing real-world systems.

I-2. Homogeneous equation:

A homogeneous second-order differential equation is a differential equation in which the dependent variable and its derivatives are combined linearly and equated to zero. Such equations typically take the form:

$$\ddot{x} + a\dot{x} + bx = 0 \qquad (1.1)$$

where a and b are constants, and x is the unknown function. The general solution to this equation is determined by solving the characteristic equation:

$$r^2+ar+b=0 \qquad (1.2)$$

Before proceeding with the solution, it is important to clarify the physical meaning of the various terms in equation (1.1) within the context of vibration theory. In the case of weak mechanical vibrations, the constants in equation (1.1) represent specific physical quantities: 2δ is the damping coefficient of the vibration, w_0^2represents the natural frequency (or proper pulsation) of the vibration.

Thus, the equation can be rewritten as:

$$\ddot{x} + 2\delta\dot{x} + w_0^2 x = 0 \qquad (1.3)$$

and the corresponding characteristic equation becomes:

$$r^2 + 2\delta r + \omega_0^2 = 0 \qquad (1.4)$$

The solution to equation (1.3) is directly related to the solutions of the characteristic equation (1.4).

Calculation of the discriminant of the quadratic polynomial (1.4)**:**

$$\boldsymbol{\Delta' = \delta^2 - \omega_0^2} \qquad \textbf{(1.5)}$$

- ✓ if $\Delta' \rangle 0$that is to say $\delta \rangle \omega_0$, equation (4) has two real nigative solutions

  $r_1 = -\delta - \sqrt{\Delta'}$ et $r_2 = -\delta + \sqrt{\Delta'}$ (1.6)

  The solution $x(t)$ to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ will be:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad (1.7)$$

Where C_1 and C_2 are two real constants, and $t$is the independent variable. The constants C_1 and C_2 are determined from the initial or boundary conditions.

- ✓ if $\Delta' = 0$that is $\delta = \omega_0$, equation (1.4) has a non-zero double real solution $r = -\delta$.

  The solution $x(t)$ to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ will be:

$$x(t) = (C_1 + C_2 t)e^{-\delta t} \text{ (8)}$$

where C_1 and C_2 are two real constants.

- ✓ if $\Delta' \langle 0$that is , $\delta \langle \omega_0$ equation (1.4) has two complex solutions $r_1 = \alpha - i\beta$ and $r_2 = \alpha + i\beta$ Or $\alpha = -\delta$ and $\beta = \sqrt{\omega_0^2 - \delta^2}$ (1.9)

  The solution $x(t)$ to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ will be:

  $x(t) = (C_1 \cos(\beta t) + C_2 \sin(\beta t))e^{-\delta t}$ Or $x(t) = (C \cos(\beta t + \varphi))e^{-\delta t}$ (1.10)

  Where C_1, C_2 and φ are real constants.

Example 1: The elastic pendulum was designed using a spring with a constant k, a mass M, and a damping device (shock absorber) characterized by a friction coefficient α, as shown in figure 1.

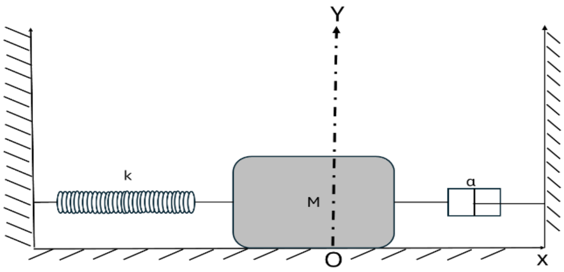


Fig.1: elastic pendulum with demping force.

This system can be described by equation (3), as will be demonstrated in this chapter. Therefore, we can write: $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$, It is important to recall that $w_0 = \sqrt{\frac{k}{m}}$:

I-2-1. Highly damped regime:

- ❖ If $\Delta' \rangle 0$ that is mean $\delta \rangle \omega_0$ (meaning the frictional force is greater than the restoring force of the spring), the solution to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ is given by: $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

- Let us consider the following numerical example: $\ddot{x} + 8\dot{x} + 7x = 0$, The corresponding characteristic equation is: $r^2 + 8r + 7 = 0$, and the discriminant of this equation is:

$$\boldsymbol{\Delta' = \delta^2 - \omega_0^2 = 16 - 7 = 9 \Rightarrow \sqrt{\Delta'} = 3, r_1 = \frac{-4-3}{1} = -7 \; and \; r_2 = \frac{-4+3}{1} = -1}$$

$$x(t) = C_1 e^{-7t} + C_2 e^{-t}$$

Using the initial conditions x(t=0s)=1cm, and $\dot{x}(t = 0s) = 0cms^{-1}$, the constants C_1 and C_2 are found to be:

$$\begin{cases} x(t = 0s) = 1cm \Rightarrow C_1 + C_2 = 1 \\ \dot{x}(t = 0s) = 0cms^{-1} \Rightarrow -7C_1 - C_2 = 0 \end{cases} \Rightarrow \begin{cases} x(t = 0s) = 1cm \Rightarrow C_1 = {}^{-1}/_{6} \\ \dot{x}(t = 0s) = 0cms^{-1} \Rightarrow C_2 = +{}^{7}/_{6} \end{cases}$$

$$x(t) = -\frac{1}{6}e^{-7t} + \frac{7}{6}e^{-t}$$

The graph of this function is shown in the following figure. In vibration physics, this system is classified as overdamped or aperiodic.

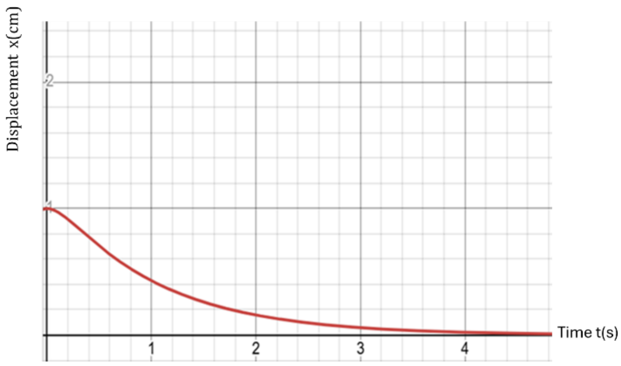


Fig.2: Variation of X as a function of time for overdamped system.

I-2-2. Critical regime:

- if $\Delta' = 0$ that is mean $\delta = \omega_0$ (the damping force is of the same order as the restoring force of the spring), the solution to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ is given by: $x(t) = (C_1 + C_2 t)e^{-\delta t}$.
- Let us consider the following numerical example: $\ddot{x} + 4\dot{x} + 4x = 0$, The corresponding characteristic equation is: $r^2 + 8r + 4 = 0$, and the discriminant of this equation is: $\boldsymbol{\Delta' = \delta^2 - \omega_0^2 = 4 - 4 = 0 \Rightarrow r_1 = r_2 = -\delta = -2} \Rightarrow x(t) = (C_1 + C_2 t)e^{-2t}$ **,** To solve this equation using the initial conditions x(0)=1 cm and $\dot{x}(0) = 0m/s$, follow these steps:

Step 1: Apply the initial condition x(0)=1 cm at t=0:

$x(0)=(C_1+C_2\cdot 0)e^0=C1$, thus, $C_1=1$.

Step 2: Apply the initial condition $\dot{x}(0) = 0m/s$

First, compute the derivative of x(t):

$$\dot{x} = \frac{d}{dt}[(C_1 + C_2 t)e^{-2t}]$$

Using the product rule:

$$\dot{x}(t) = e^{-2t}\frac{d}{dt}[(C_1 + C_2 t)] + [(C_1 + C_2 t)]\frac{d}{dt}e^{-2t}$$

$$\text{So } \dot{x}(t) = C_2 e^{-2t} - 2[(C_1 + C_2 t)e^{-2t}]$$

$$\text{At t=0: } \dot{x}(t) = C_2 - 2C_1$$

Using $\dot{x}(0) = 0m/s$ and $C_1=1$:

$C_2-2(1) =0$

$C_2=2$ than $x(t) = (1 + 2t)e^{-2t}$

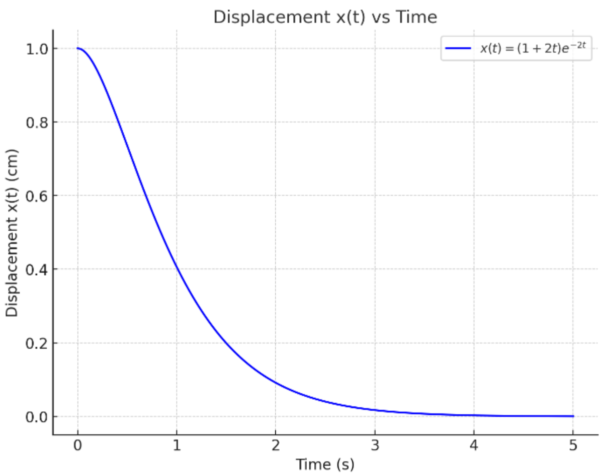


Fig.3: the displacement over time.

Fig.3 illustrates how the displacement begins at 1 cm and gradually diminishes over time, governed by the damping factor e^{-2t}. In the context of vibration physics, this system is classified as operating in the critical damping regime. Critical damping occurs when the damping is just sufficient to return the system to equilibrium without oscillation. The system returns to rest as quickly as possible without overshooting, as reflected in the smooth decay of the displacement in the figure.

I-2-3. Pseudoperiodic regime:

- if $\Delta' \langle\ 0$ that is , $\delta \langle \omega_0$ (meaning the restoring force of the spring is greater than the frictional force) in this cas the solution to the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ is given by: $x(t) = (C\,cos(\beta t + \varphi))e^{-\delta t}$.
- Let us consider the following numerical example: $\ddot{x} + 2\dot{x} + 4x = 0$, The corresponding characteristic equation is: $r^2 + 2r + 4 = 0$, and the discriminant of this equation is: $\boldsymbol{\Delta' = i\beta}$ indicating complex roots. Thus, the two complex solutions are given by: $r_1 = -\delta + i\sqrt{\omega_0^2 - \delta^2}$ and $r_2 = -\delta - i\sqrt{\omega_0^2 - \delta^2} \Rightarrow r_1 = -1 + i\sqrt{3}\ and\ r_2 = -1 - i\sqrt{3}$ so the general solution is: $x(t) = (C\,cos(\sqrt{3}t + \varphi))e^{-t}$ or $x(t) = e^{-t}(C_1 cos(\sqrt{3}t) + C_2 sin(\sqrt{3}t))$ where C_1=Ccos(φ) and C_2=Csin(φ). To determine these constantes we take in concederation the folowing initial conditions x(0)=1 cm and $\dot{x}(0) = 0m/s$.

Step 1: Evaluate x(0)

Substituting t=0: x(0)=(Ccos(φ))e^0=Ccos(φ). Setting this equal to the initial condition:

$$C\cos(\varphi)=1 \qquad (1)$$

Step 2: Evaluate $\dot{x}(0)$, first, we need to find the derivative $\dot{x}(t)$:

We have: $\dot{x}(t) = \frac{d}{dt}\left[Ccos(\sqrt{3}t + \varphi)e^{-t}\right]$ applying the product rule:

$$\dot{x}(t) = -\sqrt{3}e^{-t}\left[Csin(\sqrt{3}t + \varphi)\right] - \left[Ccos(\sqrt{3}t + \varphi)e^{-t}\right]$$

Now, evaluating at t=0:

$$\dot{x}(0) = -\sqrt{3}[Csin(\varphi)] - [Ccos(\varphi)] = 0 \qquad (2)$$

We now have two equations to solve: $\begin{cases} C\cos(\varphi) = 1 & (1) \\ \dot{x}(0) = -\sqrt{3}[Csin(\varphi)] - [Ccos(\varphi)] = 0 & (2) \end{cases}$

Step 3: Solve the system of equations

From (1), we can express C: $C = \frac{1}{cos(\varphi)}$ (3)

Substituting (3) into (2): $-\sqrt{3}\left[\frac{1}{cos(\varphi)} sin(\varphi)\right] - \left[\frac{1}{cos(\varphi)} cos(\varphi)\right] = 0$

Multiplying through by cos(φ):

$$-\sqrt{3}[sin(\varphi)] - [1] = 0$$

Rearranging gives: $\sqrt{3}[sin(\varphi)] = -1$.Thus: $sin(\varphi) = \frac{-1}{\sqrt{3}} \Rightarrow \varphi = -\frac{\pi}{6}$ (in the range of sine).

Tep 4: Calculate C: Using $cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, from (3): $C = \frac{2}{\sqrt{3}}$

So the general solution has this form: $x(t) = (\frac{2}{\sqrt{3}}cos(\sqrt{3}t - \frac{\pi}{6}))e^{-t}$

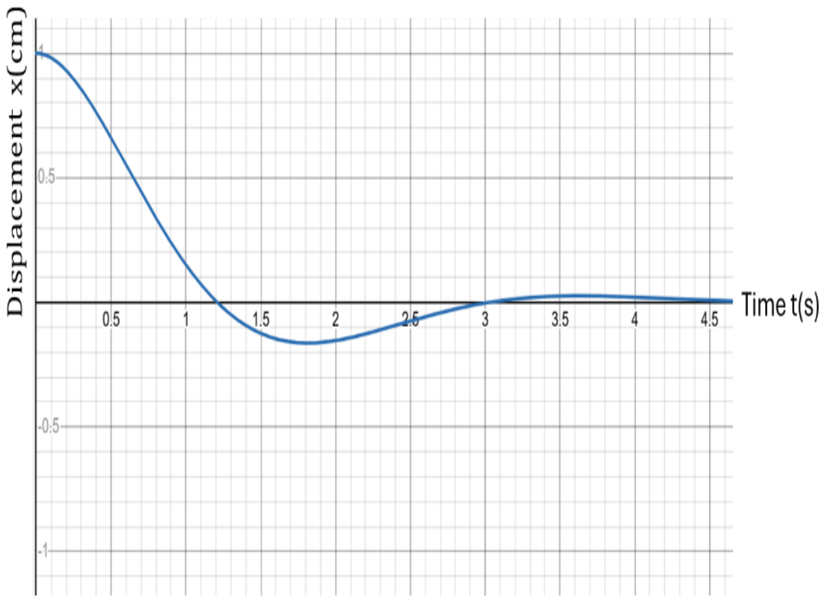


Fig.4: damped oscillations

For small t: The solution starts with an initial displacement of 1 cm and oscillates. The amplitude of the oscillation is initially large, but the exponential decay e^{-t} quickly reduces the size of the oscillations. As t→∞: The damping causes the oscillations to eventually fade out. The displacement x(t) asymptotically approaches zero as the term e^{-t} decays to zero (Fig.4). In the context of vibration physics, this system is classified as operating in the Pseudoperiodic regime.

I-2-4. Summary of three regimes:

- ✓ In simple terms, the general equation of motion for a vibrating system, such as a mass-spring-damper setup (see Fig.1), is often modeled by a homogene second-order differential equation. For undisturbed or "free" vibrations, the equation is homogeneous, meaning no external forces act on the system.

- ✓ **Overdamped ($\delta \rangle \omega_0$):** Slow, non-oscillatory return to equilibrium.

- ✓ **Critically Damped ($\delta=\omega_0$)**: Fastest, non-oscillatory return to equilibrium.
- ✓ **Underdamped (Pseudoperiodic) ($\delta < \omega_0$)**: Oscillatory motion with exponentially decaying amplitude; the decay is slower than critical damping
- ✓ Each regime describes how the damping influences the system's ability to return to equilibrium, which is essential in applications like mechanical engineering, electronics (RLC circuits), and control systems where stable, rapid, or smooth behavior is needed.

1-3. Equation with second member:

1-3-1. Introduction:

In the study of vibrations in physics, differential equations with a second member, often referred to as non-homogeneous differential equations, play a critical role in describing systems subject to external forces. These types of equations extend the analysis beyond natural oscillations by incorporating external influences, such as driving forces or damping, that impact the system's behavior. In this cas the system is described by a non-homogeneous differential equation:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = F(t) \qquad (1.11)$$

F(t) is the external force (the "second member").

1-3-2. General solution:

We assume $x_G(t) = x_H(t) + x_P(t)$ the general solution of equation (1.11) where $x_H(t)$ is the solution of the homogeneous equation $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ and $x_P(t)$ is the particular solution of equation (11). To determine the overall solution, we need to find both the homogeneous and particular solutions. The form of the particular solution $x_P(t)$ of eq (1.11) depends on the form of the right-hand side (*F(t)*).

a- If $F(t) = P_n(t)\mathrm{e}^{\alpha t}$ where $P_n(t)$ is a polynomial of degree **n**

- ✓ If α is not a solution of the characteristic equation (2) the solution $x_P(t)$ sought will be of the form: $x_p(t) = Q_n(t)\mathrm{e}^{\alpha t}$ where $Q_n(t)$ is a polynomial of degree **n**
- ✓ If α is a simple solution of the characteristic equation (2) the solution will be written as follows: $x_p(t) = t\ Q_n(t)\mathrm{e}^{\alpha t}$ where $Q_n(t)$ is a polynomial of degree **n**

- ✓ If α is a double solution of the characteristic equation (III) the solution $x_P(t)$ will be written as follows: $x_p(t)=t^2\ Q_n(t)\mathrm{e}^{\alpha t}$ where $Q_n(t)$ is a polynomial of degree **n**

b- If $F(t)=P(t)\mathrm{e}^{\alpha t}\cos(\beta t)+Q(t)\mathrm{e}^{\alpha t}\sin(\beta t)$ where $P(t)$ and $Q(t)$ are two polynomials.

- ✓ If $\alpha+i\beta$ is not a solution of (2), the particular solution of eq (11) is written $x_P(t)=V(t)\mathrm{e}^{\alpha t}\cos(\beta t)+U(t)\mathrm{e}^{\alpha t}\sin(\beta t)$, $V(t)$ and $U(t)$ being polynomials whose degree is equal to the highest degree of $P(t)$ and $Q(t)$.
- ✓ If $\alpha+i\beta$ is a solution of (2), the particular solution of eq (11) is written as $x_P(t)=t\left[V(t)\mathrm{e}^{\alpha t}\cos(\beta t)+U(t)\mathrm{e}^{\alpha t}\sin(\beta t)\right]$, $V(t)$ and $U(t)$ being polynomials whose degree is equal to the highest degree of $P(t)$ and $Q(t)$.

c- Special case: $P(t)=M$ et $Q(t)=N$, $\alpha=0$ Or M and N are constants.

$F(t)=M\cos(\beta t)+N\sin(\beta t)$ **.**

- ✓ If $i\beta$ is not a solution of (2), the particular solution of eq (11) is written $x_P(t)=A\cos(\beta t)+B\sin(\beta t)$**,** A and B being constants to be determined.
- ✓ If $i\beta$ is a solution of (2), the particular solution of eq (11) is written as $x_P(t)=t\left[A\cos(\beta t)+B\sin(\beta t)\right]$, A and Bbeing constants to be determined.

Noticed :

The form $F(t)=M\cos(\beta t)+N\sin(\beta t)$ is the same with the form $F(t)=C\cos(\beta t+\varphi)$, in this case the solution is of the form:

- ✓ If $i\beta$ is not a solution of (2), the particular solution of eq (11) is written $x_P(t)=A\cos(\beta t+\psi)$**,** A and ψ being constants to be determined.
- ✓ $i\beta$ is a solution of (2), the particular solution of eq (11) is written as: $x_P(t)=t[A\,cos(\beta t+\psi)]$, A and ψ being constants to be determined.

Example2:

Solve the following differential equations:

1) $\ddot{x}+9x=(t-2)e^t$. (a)
2) $\ddot{x}+7\dot{x}+6x=(t-2)e^t$. (b)

solution:

Equation (a):

The analogy of equation (3) with the differential equation of a mechanical vibration allows us to say whether this system is forced, damped, forced damped or harmonic.

We have:

$\delta = 0$ et $F(t) = (t-2)e^{t}$ And $\omega_0^2 = 9$ So our system is in forced mode.

The homogeneous equation of equation (1) represents a system in harmonic regime (undamped and unforced)

The resolution of equation (1) is $x_{1G}(t) = x_{1H}(t) + x_{1P}(t)$.

The homogeneous solution $x_{1H}(t)$:

The characteristic equation of (1) is: $r^2 + 9r = 0$

The resolutions of characteristic eq are $r_1 = 3i$ and $r_2 = -3i$, we have two complex solutions where $\alpha = 0$ et $\beta = 3$ therefore the homogeneous solution will be :

$x_H(t) = C_1\cos(3t) + C_2\sin(3t)$ ou $x_H(t) = C\cos(3t + \varphi_0)$, the constants are a function of the initial conditions.

The particular solution $x_{1P}(t)$:

We have: $F(t) = (t-2)e^{t}$ that is to say that $F(t) = P_n(t)\,e^{\alpha t}$ where $P_n(t) = t-2$ and $\alpha = 1$ therefore our particular solution will be of the form: $x_p(t) = Q_n(t)\,e^{\alpha t}$ because α it is not a solution of the characteristic equation.

Let's look $x_{1P}(t) = (at + b)e^{t}$ after treatment we found:

$a = 0.1$ et $b = -0.22$ from where $x_{1P}(t) = (0.1\,t - 0.22)e^{t}$.

SO $x_{1G}(t) = C_1\cos(3t) + C_2\sin(3t) + (0.1\,t - 0.22)e^{t}$.

Equation (b):

$$\ddot{x} + 7\dot{x} + 6x = (t-2)e^{t}$$

We have:

$\delta = 3.5$ and $F(t) = (t-2)e^{t}$ And $\omega_0^2 = 9$ So our system is in a forced damped regime.

Let us determine $x_{2H}(t)$:

The characteristic equation is : $r^2 + 7r + 6 = 0$.

$\Delta^{'} = 3.5^2 - 6 = 6.25 \rangle\ 0 \Rightarrow \sqrt{\Delta^{'}} = 2.5$

zero $r_1 = -9.5$ and $r_2 = -4.5$ real solutions , so equation (b) has a solution of the form

$x_H(t) = C_1 e^{-9.5\,t} + C_2 e^{-4.5\,t}$

Let us determine $x_{2P}(t)$:

We have: $F(t) = (t-2)e^{t}$ that is to say that $F(t) = P_n(t)\mathrm{e}^{\alpha t}$ where $P_n(t) = t - 2$ and $\alpha = 1$

therefore our particular solution will be of the form: $x_p(t) = Q_n(t)\mathrm{e}^{\alpha t}$ because α it is not a solution of the characteristic equation.

Let's look $x_{1P}(t) = (\mathrm{a}t + b)e^{t}$ after treatment we found.

$a = \frac{1}{14}$ et $b = -\frac{37}{196}$ from where $x_{2P}(t) = (\frac{1}{14}\ t - \frac{37}{196})\ e^{t}$ and so

$x_{2G}(t) = C_1 e^{-9.5\,t} + C_2 e^{-4.5\,t} + (\frac{1}{14}\ t - \frac{37}{196})\ e^{t}$, The constants are a function of the initial conditions.

Test your comprehension

Problem 1:

A mass m=2 is attached to a spring with a spring constant k=8 N. It is displaced 0.1 from equilibrium and released without any damping or external force (see Fig.5).

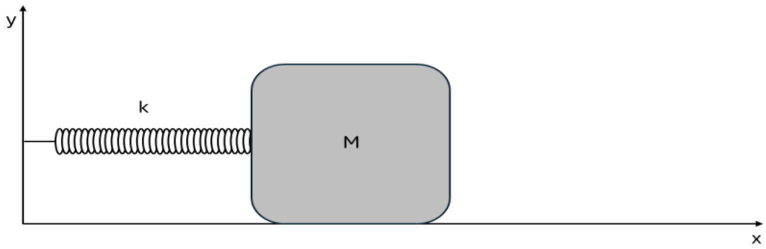


Fig.5: Ideal spring-mass system (harmounic oscillator)

- (a) Write the differential equation for the motion.
- (b) Solve the equation for the displacement x(t).
- (c) Find the period and frequency of the oscillations.

Problem 2:

A damping force $-b\frac{dx}{dt}$, with b=3 Ns/m, acts on a mass-spring system where m=2 kg and k=8 N/m. The system starts at x(0)=0.1 m with v(0)=0 m/s.

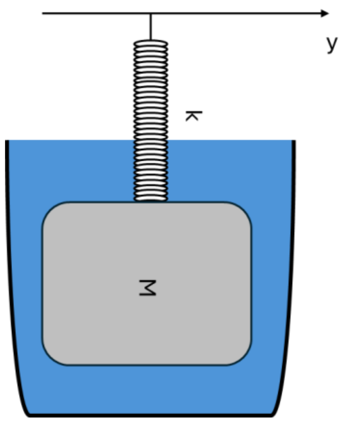


Fig.6: spring-mass system with a damping force

- (a) Classify the type of damping (overdamped, underdamped, or critically damped).
- (b) Solve the differential equation for x(t).
- (c) Plot x(t) for the three damping cases by varying b_1=3Ns/m, b_2=8Ns/m, b_3=12Ns/m.

Problem 3:

A forced oscillator satisfies the equation:

$$m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = F_0\cos(wt)$$

where m=1 kg, α=0.5 Ns/m, k=4 N/m, F_0=1 N, and ω is variable.

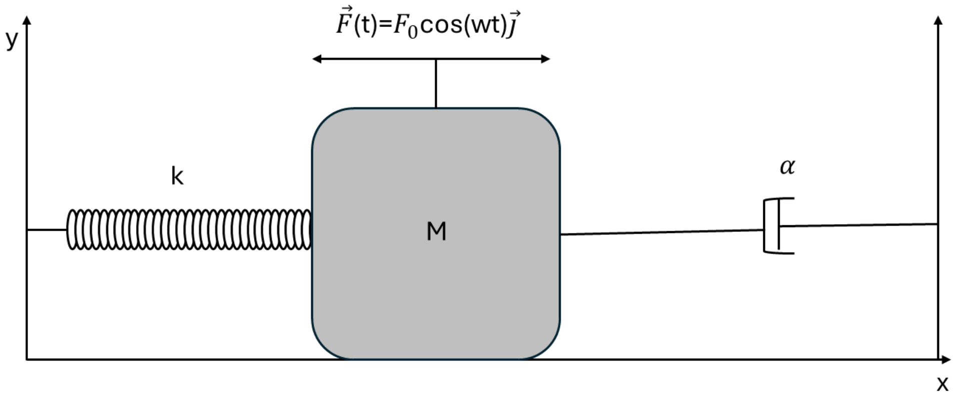


Fig.7: Forced Oscillator

- (a) Find the steady-state solution for x(t).
- (b) Determine the resonance frequency of the system.
- (c) Plot the amplitude of the steady-state solution as a function of ω.

Solutions

Problem 1:

Summary solution:

- The differential equation is $\ddot{x} + 4x = 0$.
- The displacement is $x(t)=0.1\cos(2t)$.
- The period is approximately 3.14 seconds and the frequency is approximately 0.318 Hz.

Problem 2:

(a) Classify the type of damping

The equation of motion for a damped harmonic oscillator is:

$$m\ddot{x}+b\dot{x}+kx=0$$

Given:

- Mass: m=2 kg
- Damping coefficient: b=3 Ns/m
- Spring constant: k=8 N/m

The discriminant of the characteristic equation determines the type of damping:

$$\zeta = \frac{b}{2\sqrt{mk}}$$

$$\zeta = 0.375$$

Since $\zeta<1$, the system is **underdamped**.

(b) Solve the differential equation for x(t)

The general solution for an underdamped system is:

$$x(t) = Ae^{-\zeta w_0 t}\cos\,(w_D t + \varphi)$$

Damped angular frequency (ω_D): $w_D = w_0\sqrt{1-\zeta^2} = 2\sqrt{1-(0.375)^2} \approx 1.85\ rads/s$

Solve for A and φ using initial conditions:

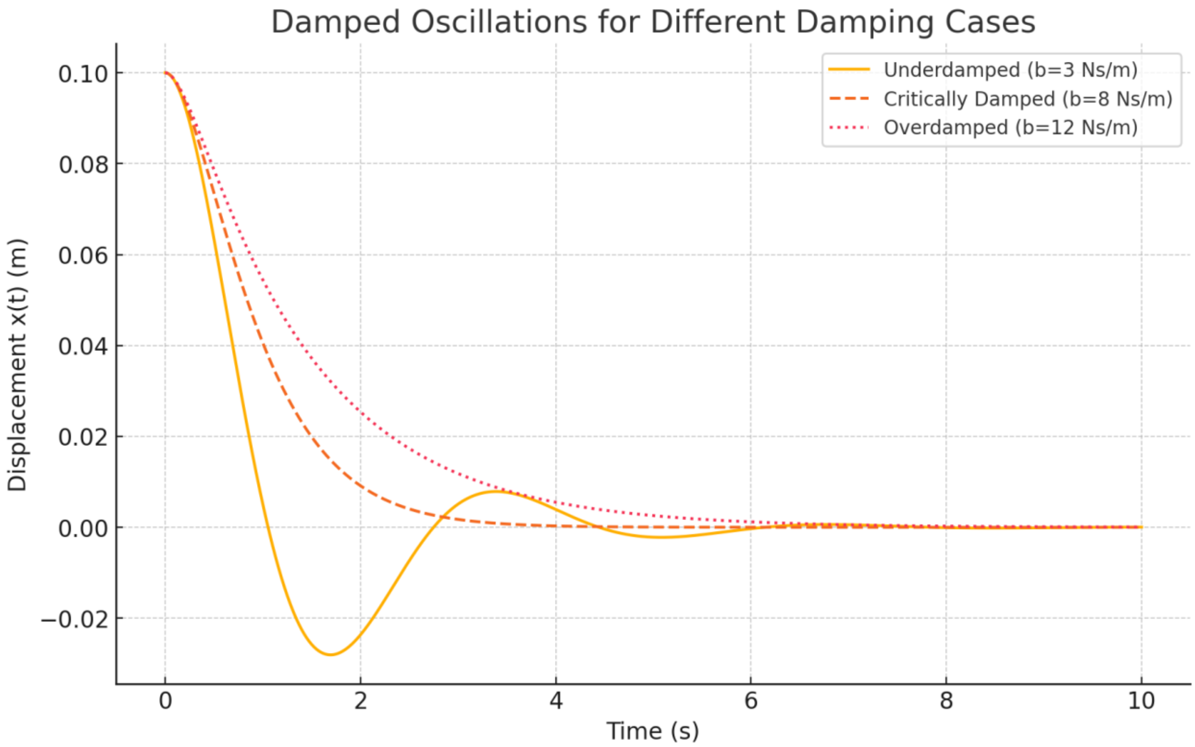
Final Solution: The displacement x(t) is:

$$x(t)=0.11e^{-0.75t}\cos(1.85t-0.38)$$

c) Plot x(t)x(t)x(t) for the three damping cases by varying bbb

We will compute x(t) for three cases:

1. **Underdamped**: b=3 Ns/m.
2. **Critically damped**: $b=2\sqrt{mk}$=8 Ns/m
3. **Overdamped**: b=12 Ns/m.



Problem 3:

The differential equation for the forced oscillator is:

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(wt) \; or \; \ddot{x} + 2\delta\dot{x} + w_0^2 x = \frac{F_0}{m}\cos(wt)$$

(a) Find the steady-state solution x(t):

The steady-state solution for a forced oscillator is given by:

$$x(t)=X\cos(\omega t-\phi)$$

where:

- X is the amplitude of oscillation,
- ϕ is the phase lag.

1. Amplitude X: $X = \frac{F_0/m}{\sqrt{({w_0}^2-w^2)^2+4\delta^2 w^2}}$
2. Phase lag ϕ: $\phi = -\text{actg}\frac{2\delta w}{({w_0}^2-w^2)}$

(b) Resonance Condition

Resonance occurs when the system's response (amplitude X) is maximized. The amplitude X depends on the denominator of the formula: $D(w) = \sqrt{({w_0}^2 - w^2)^2 + 4\delta^2 w^2}$

For X to be maximized, D(ω) must be minimized.

Minimize D(ω):

The denominator consists of two terms:

${(w_0}^2 - w^2)^2$: Represents the detuning of the forcing frequency from the natural frequency.

$4\delta^2 w^2$: Represents the influence of damping.

The minimum of D(ω) occurs when the first term and second term balance optimally.

Differentiate $D^2(\omega)$ with respect to ω to find the frequency ω_r where D(ω) is minimized. Setting the derivative $\frac{d}{dw}D^2(w) = 0$:

$$\frac{d}{dw}\left[({w_0}^2 - w^2)^2 + 4\delta^2 w^2\right] = 0$$

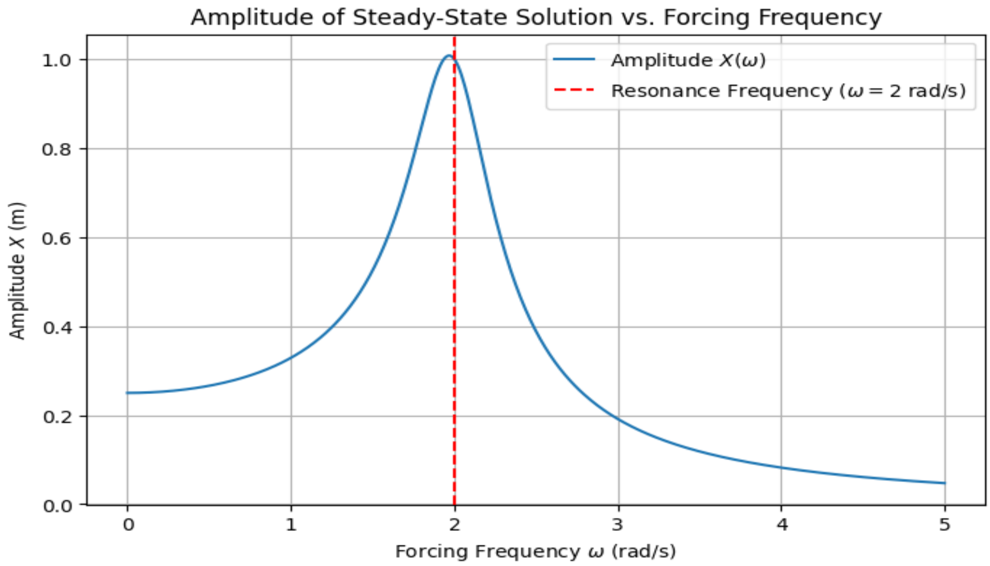
$$w[-4(w_0^2 - w^2) + 8\delta^2] = 0$$

Since ω=0 is not meaningful for resonance, solve:

$$w_r = \sqrt{w_0^2 - 2\delta^2}$$

Special Cases

1. **Undamped System (δ=0):** Resonance occurs at $\omega_r=\omega_0$, and the amplitude becomes infinite in theory.
2. **Heavily Damped System:** For large δ where $\delta > \frac{w_0}{\sqrt{2}}$, the resonance peak becomes broader and less pronounced.



Amplitude of Steady-State Solution vs. Forcing Frequency
Amplitude $X(\omega)$
Resonance Frequency ($\omega = 2$ rad/s)
Amplitude X (m)
Forcing Frequency ω (rad/s)
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5

Chapter 2: Free oscillations of one-degree-of-freedom systems

2-1. Introduction:

This chapter offers a detailed examination of free oscillations in one-degree-of-freedom systems, a fundamental concept in the field of mechanical vibrations. One-degree-of-freedom systems, defined by motion restricted to a single independent direction or degree of freedom, serve as simplified yet effective models for capturing the essential dynamics of oscillatory behavior. The analysis of free oscillations, occurring in the absence of external forces, is critical for determining key system parameters such as natural frequency and mode shapes, both of which play a significant role in influencing the system's dynamic response and overall stability. By exploring the governing equations and the system's response characteristics, this chapter seeks to develop a theoretical foundation essential for more advanced topics in vibration analysis. Prior to delving into these concepts, it is advisable to formalize the discussion within the framework of Lagrangian mechanics.

2-2. Lagrange Formalism:

2-2-1. Generalized coordinates and degrees of freedom:

To determine the position of a system of ***N*** material points in space, it is necessary to specify ***N*** radius vectors, which correspond to ***3N*** coordinates**.** In general, ***the number of independent quantities that must be given to determine unequivocally the position of a system is called the number of degrees of freedom of the system***. In this present case, this number is equal to 3N. These quantities are not necessarily the Cartesian coordinates of the point, and depending on the conditions of the problem, the choice of another coordinate system may be more convenient.

Any quantities $\mathbf{q_1, q_2, ..., q_s}$ completely characterizing the position of the system (with **S** degrees of freedom) are called generalized coordinates.

The vibrational approach is considerably simplified to the extent that a set of independent geometric variables can be found, variables that can vary independently of each other and that allow to represent all configurations compatible with the system's bonds. The number of degrees of freedom is equal to the number of coordinates that represent the position of [**N masses** x (three demonstrations of space)] minus the number of bonds **n** .

Then the number of degrees of freedom $$\mathbf{S = 3N-n} \qquad \mathbf{(2.1)}$$

Where **N** : number of material points (or number of masses).

N : number of bonds or (geometric constraints).

Example 1:

A particle of mass m is forced to move along a parabolic trajectory ($y=ax^2$) (Figure I).

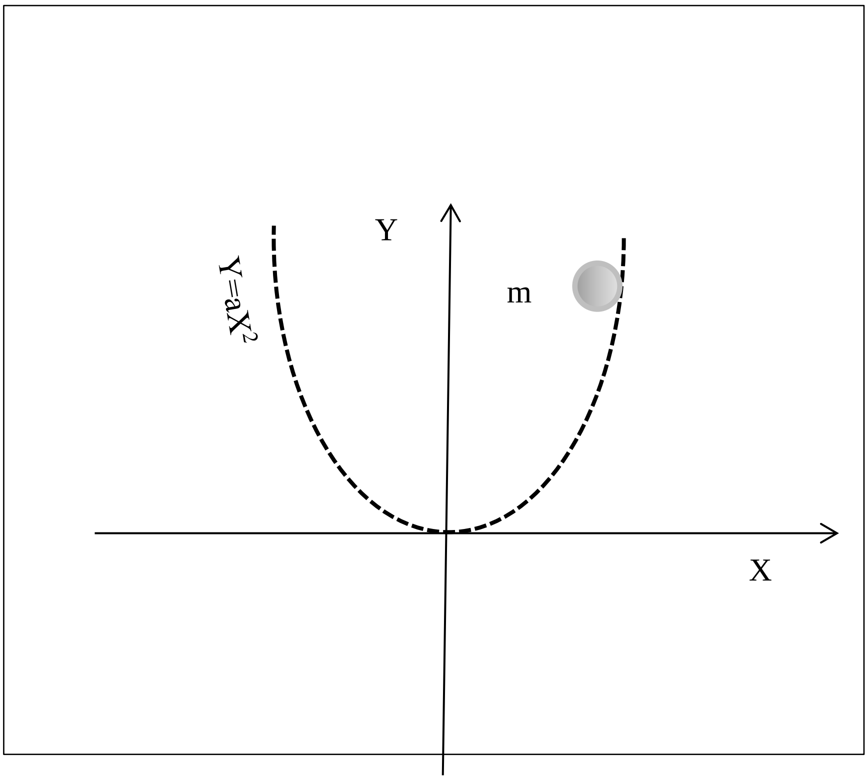


Fig. 1

Number of bonds n: $\begin{cases} y = ax^2 \\ z = 0 \end{cases}$; two bonds and the number of masses 1.

So number of degrees of freedom S = 3 (1) – 2 =1.

If we choose Y as the generalized coordinate, this choice would be wrong because:

$Y= X^2 = (-X)^2$, So, we cannot know whether the mass is in the positive or negative side of the X axis; but we can consider X as a generalized coordinate.

2-2-2. Introduction to Lagrange's equations « Lagrange equations for a particle»:

Consider the special case of a particle with one degree of freedom. A variable q must be chosen to locate its position. This variable is called a generalized coordinate. The position vector of the particle can be expressed $\vec{r}$as a function of the generalized coordinate q by the relation:

$$\vec{r} = \vec{r}\,(q) \qquad (2.2)$$

Let be $\vec{F}$the resultant of all the forces acting on the particle. The fundamental relation of dynamics is written:

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} \qquad (2.3)$$

Where $\vec{v} = \frac{\partial \vec{r}}{\partial t}$ is the velocity of the particle.

Let δW be the work provided by the force $\vec{F}$during an infinitesimal displacement δ $\vec{r}$:

$$\delta W = \vec{F} . \delta\vec{r} \qquad (2.4)$$

The infinitesimal displacement δ $\vec{r}$can be written as a function of the variation δq of the coordinate generalized q:

$$\delta \vec{r} = \frac{\partial \vec{r}}{\partial q}\delta q \qquad (2.5)$$

In this case the work δW can be given the form:

$$\delta w = \vec{F}.\frac{\partial \vec{r}}{\partial q}\delta q \qquad (2.6)$$

The generalized conjugate force of q, or q-component of the force, is called the quantity F $_q$ defined by:

$$F_q = \frac{\delta w}{\delta q} = \vec{F} \cdot \frac{\partial \overrightarrow{r_q}}{\partial q} \qquad (2.7)$$

Therefore δW is written:

$$\delta W = F_q \delta q \qquad (2.8)$$

Taking into account the fundamental relationship of dynamics, this expression can also be written:

$$\delta W = m\frac{d\vec{v}}{dt} \cdot \frac{\partial \vec{r}}{\partial q}\delta q \qquad (2.9)$$

On the other hand : $\frac{d}{dt}\left[\vec{v} \cdot \frac{\partial \vec{r}}{\partial q}\right] = \frac{d\vec{v}}{dt} . \frac{\partial \vec{r}}{\partial q} + \vec{v} \cdot \frac{d}{dt}\left[\frac{d\vec{r}}{dq}\right]$ (2.10)

Knowing that:

$$\frac{d}{dt}\left[\frac{\partial \vec{r}}{\partial q}\right] = \frac{\partial}{\partial q}\left[\frac{\partial \vec{r}}{\partial t}\right] = \frac{\partial \vec{v}}{\partial q} \qquad (2.11)$$

We obtain:

$$\frac{d\vec{v}}{dt} \cdot \frac{\partial \vec{r}}{\partial q} = \frac{d}{dt}\left[\vec{v} \cdot \frac{\partial \vec{r}}{\partial q}\right] - \vec{v}\frac{\partial \vec{v}}{\partial q} \qquad (2.12)$$

The velocity vector $\vec{v}$, can also be written: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{\frac{d\vec{r}}{dt}}{\frac{dq}{dt}} = \frac{\partial \vec{v}}{\partial \dot{q}}$ (2.13)

Hence the relationship:

$$\frac{\partial \vec{r}}{\partial q} = \frac{\partial \vec{v}}{\partial \dot{q}} \qquad (2.14)$$

And

$$\frac{d\vec{v}}{dt} \cdot \frac{\partial \vec{r}}{\partial q} = \frac{d}{dt}\left[\vec{v} \cdot \frac{\partial \vec{r}}{\partial \dot{q}}\right] - \vec{v}\frac{\partial \vec{v}}{\partial q} \qquad (2.15)$$

Knowing that : $$\frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}v^2\right] = \frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}\vec{v}\cdot\vec{v}\right] = \vec{v}\cdot\frac{\partial \vec{v}}{\partial \dot{q}} \qquad (2.16)$$

and that:

$$\frac{\partial}{\partial q}\left[\frac{1}{2}v^2\right] = \frac{\partial}{\partial q}\left[\frac{1}{2}\vec{v}\cdot\vec{v}\right] = \vec{v}\cdot\frac{\partial \vec{v}}{\partial q} \qquad (2.17)$$

We obtain: $$\frac{d\vec{v}}{dt}\cdot\frac{\partial \vec{r}}{\partial q} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}v^2\right]\right] - \frac{\partial}{\partial q}\left[\frac{1}{2}v^2\right] \qquad (2.18)$$

We now reformulate the expression for work δW, taking into consideration equation (2.18):

$$\delta W = m\left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}v^2\right]\right] - \frac{\partial}{\partial q}\left[\frac{1}{2}v^2\right]\right\}\delta q \qquad (2.19)$$

Let $T = \frac{1}{2}mv^2$ denote the kinetic energy of the mass m (assuming m is time-independent):

$$\delta W = \left\{\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q}\right\}\delta q \qquad (2.20)$$

We finally obtain the two equivalent expressions (eq 2.8 and eq 2.20) of the work δW :

$$\left\{\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q}\right\}\delta q = F_q\delta q \qquad (2.21)$$

We deduce the d'Alembert equation for a system with one degree of freedom:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q} = F_q \qquad (2.22)$$

2-2-3. Case of conservative systems

In conservative systems, the applied force to the system derives from a potential U and it is written as:

$$F_q = -\frac{\partial U}{\partial q} \qquad (2.23)$$

Lagrange's equation then becomes:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q} = -\frac{\partial U}{\partial q} \qquad (2.24)$$

Generally, the potential energy U does not depend on the velocity, which is to say that $\frac{\partial U}{\partial \dot{q}} = 0.$

The Lagrange equation can then be written:

$$\frac{d}{dt}\left[\frac{\partial(T-U)}{\partial\dot{q}}\right]-\frac{\partial(T-U)}{\partial q}=0 \qquad (2.25)$$

We introduce the Lagrange function (or Lagrangian of the system) which is the difference between the kinetic energy and the potential energy:

$$L=T-U \qquad (2.26)$$

Hence the form of the Lagrange equation in the case of a conservative system:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}}\right]-\frac{\partial L}{\partial q}=0 \qquad (22.7)$$

1.1.3 Case of friction forces with velocity-dependent

Lagrange equation

Consider a physical situation in which the particle is subjected to viscosity friction forces whose resultant is of the form: $\vec{f}=-\alpha\vec{v}$ (2.28)

To calculate the corresponding generalized force $\boldsymbol{f_q}$, we use the definition from the previous paragraph (see eq 2.7) :

This last expression can be put in the form: $f_q=\vec{f}.\frac{\partial\vec{r}}{\partial q}=-\alpha\left[\frac{\partial\vec{r}}{\partial q}\right]^2\frac{\partial q}{\partial t}$ (2.29)

Reformulate the eq 2.29 as : $f_q=-\beta\dot{q}$ (2.30)

where: $\beta=\left[\frac{\partial\vec{r}}{\partial q}\right]^2$ (2.31)

If in addition to the forces that derive from a potential there are viscosity friction forces, Lagrange's equation is written:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{q}}\right]-\frac{\partial T}{\partial q}=F_{U,q}+f_q \qquad (2.32)$$

Where $F_{U,q}=-\frac{\partial U}{\partial q}$ represents the forces that derive from a potential. Hence:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}}\right]-\frac{\partial L}{\partial q}=-\beta\dot{q} \qquad (2.33)$$

Dissipation function

Let us calculate the work $\boldsymbol{\delta W_f}$ provided by the friction force during a time interval $\boldsymbol{\delta t}$ for a displacement $\delta\vec{r}$:

$$\delta W_f=\vec{f}.\delta\vec{r}=-\alpha v^2\delta t \qquad (2.34)$$

The amount of heat $\boldsymbol{\delta Q}$ gained by the system is such that:

$$P_d=\frac{\delta Q}{\delta t} \quad \text{and} \quad \delta Q=\alpha v^2\delta t \qquad (2.35)$$

Let the power dissipated by frictional forces in the form of heat be:

$$P_d = \alpha v^2 \qquad (2.36)$$

This dissipated power can be expressed as a function of $\dot{q}$, by:

$$P_d = \alpha \left[\frac{d\vec{r}}{dt}\right]^2 = \alpha \left[\frac{\partial \vec{r}}{\partial q}\frac{\partial q}{\partial t}\right]^2 = \beta \dot{q}^2 \qquad (2.37)$$

By definition, the dissipation function is equal to half the dissipated power:

$$D = \frac{1}{2}P_d = \frac{1}{2}\beta\dot{q}^2 \qquad (2.38)$$

The q-component $\boldsymbol{f_q}$ of the friction force can then be written:

$$f_q = -\frac{\partial D}{\partial \dot{q}} \qquad (2.39)$$

The Lagrange equation is then written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) = 0 \qquad (2.40)$$

Example 2: Undamped Oscillations 'Ideal System'

Consider Example 1 from Chapter 1, where an elastic pendulum is modelled with a spring of constant k and a mass M. However, in this case, we neglect the effects of frictional forces, treating it as an ideal, "perfect system ", as illustrated in Fig. 2.

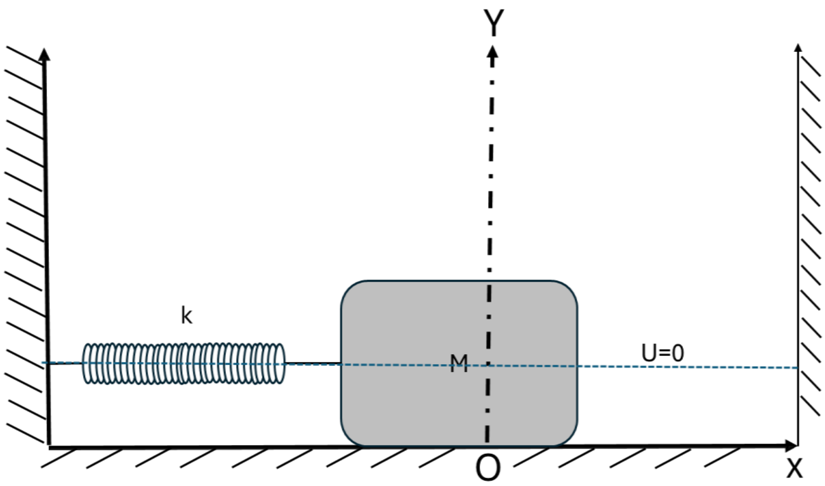


Fig.2: elastic pendulum without demping force.

1- Calculate the degrees of freedom of the system.
2- Determine the kinetic and potential energy, and subsequently derive the Lagrangian of the system.

3- Formulate the differential equation of motion and determine the natural frequency of the system.

Solution:

1- degrees of freedom of the system:

From eq (1) we have: **S = 3N-n ,** in this cas N=1 and $n = \begin{cases} z = 0 \\ y = const \end{cases}$ $so\ n = 2$ as a resulte

$\mathrm{S} = 3(1) - 2 = 1$ Thus, this system possesses only one degree of freedom.

2- The kinetic, potential energy, and the Lagrangian of the system:

The kinetic energy:

In the case of a system with one degree of freedom, consisting of a mass m whose position is identified by the generalized coordinate q, the kinetic energy is written:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{\partial \vec{r}}{\partial t}\right]^2 = \frac{1}{2}m\left[\frac{\partial \vec{r}}{\partial q}\frac{\partial q}{\partial t}\right]^2 = \frac{1}{2}m\left[\frac{\partial \vec{r}}{\partial q}\right]^2 \dot{q}^2$$

The kinetic energy of a one-degree-of-freedom system is a function of ***q*** and $\dot{q}$. It can be written in the form:

$$T = \frac{1}{2}a(q)\dot{q}^2$$

where $a\ (\ q\ )$ is a function of the generalized coordinate q , defined in the case studied by:

$$a(q) = m\left[\frac{\partial \vec{r}}{\partial q}\right]^2$$

By expanding a(q) to the second order around q=0, the kinetic energy T(q,$\dot{q}$) can be expressed as:

$$T(q,\dot{q}) = \frac{1}{2}\left[a(0) + \frac{\partial a}{\partial q}\bigg|_{q=0} q + \frac{1}{2}\frac{\partial^2 a}{\partial^2 q}\bigg|_{q=0} q^2 + ©\right]\dot{q}^2$$

Limiting the approximation to the second order of velocity $\dot{q}$, we obtain:

$$T = \frac{1}{2}a_0\dot{q}^2$$

where a_0 is a constant equal to $a\ (0)$.

Returning to our example, since x is the generalized coordinate, the kinetic energy is given by:

$$T = \frac{1}{2}a(x)\dot{x}^2$$

We then have:

$a(x) = m\left[\frac{\partial \vec{r}}{\partial x}\right]^2$ as $\vec{r} = x\vec{\imath} + y\vec{\jmath}$ with y being constant and independent of x.

Consequently: a (0) = m.

Thus, the kinetic energy simplifies to the familiar form: $\boldsymbol{T = \frac{1}{2}m\dot{x}^2}$

Potential energy

The oscillations occur around the stable equilibrium position $q = q_{eq}$, which is characterized by the condition:

$$\left.\frac{\partial U}{\partial q}\right|_{q=q_{eq}} = 0$$

For small deviations from the equilibrium position, it is possible to expand the potential energy function $U(q)$ using a Taylor series around $q = q_{eq}$. Neglecting terms of order higher than q^2, we obtain:

$$U(q) = U(0) + \left.\frac{\partial U}{\partial q}\right|_{q=q_{eq}} q + \frac{1}{2}\left.\frac{\partial^2 U}{\partial^2 q}\right|_{q=q_{eq}} q^2 + ©$$

Since $q = q_{eq}$ corresponds to the minimum of U(q), we know that:

$$\left.\frac{\partial U}{\partial q}\right|_{q=q_{eq}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 U}{\partial^2 q}\right|_{q=q_{eq}} > 0$$

By choosing the origin of potential energy such that $U(q_{eq}) = 0$, the potential energy near the equilibrium position can be expressed in its quadratic form as:

$$U(q) \approx \frac{1}{2}b_0 q^2$$

With $b_0 = \frac{\partial^2 U}{\partial^2 q^2}\Big|_{q=q_{eq}}$

So b_0 is a positive constant, reflecting the curvature of the potential energy at the equilibrium.

Returning to our example: Since the center of mass is at the same horizontal level as the origin of potential energy, the gravitational potential energy is zero. Therefore, the total potential energy of the system is purely due to the elastic energy of the spring:

$$U(q) = \frac{1}{2} b_0 x^2$$

where $b_0 = \frac{\partial^2 U}{\partial^2 x^2}\Big|_{q=0} = k$, with k being the spring constant.

As a result, the potential energy takes the form: $\boldsymbol{U(q) = \frac{1}{2} k x^2}$

Lagrangian of the system:

As metioned in the eq (26): $\mathrm{L = T - U} = \frac{1}{2}(m\dot{x}^2 - kx^2)$

Differential equation of motion and the natural frequency of the system:

Usig the eq (27), we have:

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + w_0^2 x = 0,\ w_0^2 = \frac{k}{m}$ is the natural frequency of the system

This differential equation corespends the simple harmonic oscillator, where the solution takes this fome:

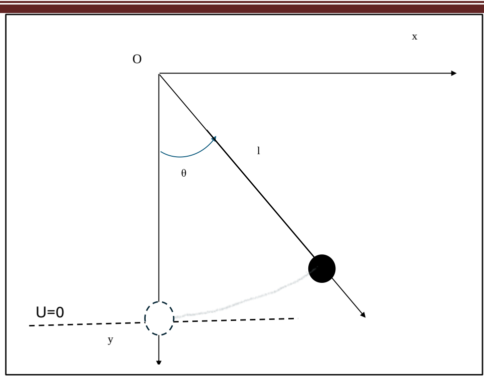
$$x(t) = x_0 cos(w_0 t + \varphi)$$

Fig.3: the graph illustrating the curve of the simple harmonic motion

Exercise:

Consider the system of a simple pendulum opposite.

- We have one degree of freedom.
- We choose θ as the generalized coordinate.



- The displacement vector $\vec{r} = l\,\vec{u}_r = l\,\sin(\theta)\,\vec{i} + l\,\cos(\theta)\,\vec{j}$

1. Determine the kinetic and potential energy, and subsequently derive the Lagrangian of the system.
2. Formulate the differential equation of motion and determine the natural frequency of the system.

Free oscillations of damped systems with one degree of freedom:

In this case, we will take into account the friction forces which are at the origin of the loss of mechanical energy of the system in the form of heat, while limiting ourselves however to the simple case where the losses are due to viscous friction for which the friction forces, which oppose the movement, are proportional to the speed.

1. **Lagrange equation for dissipative systems:**

Let us recall the Lagrange equation associated with a system with one degree of freedom whose evolution over time is reduced to the study of the generalized coordinate q

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = F_q$$

Fq represents the component along q of the resultant of the generalized forces which do not derive from a potential.

We are interested in the special case of friction forces defined by the generalized force

$$F_q = f_q = -\alpha\dot{q} = -\frac{\partial D}{\partial \dot{q}} \text{ Or } \quad D = \frac{1}{2}\alpha\dot{q}^2$$

where α is a positive real constant.

The Lagrange equation is then written in this case:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]-\frac{\partial L}{\partial q}=-\frac{\partial D}{\partial \dot{q}}=-\alpha\dot{q}$$

In the case of low amplitude oscillations, the Lagrange function was written in the form:

$$L=\frac{1}{2}a_0\dot{q}^2-\frac{1}{2}b_0q^2$$

The differential equation of motion is then written:

$$a_0\ddot{q}+b_0q+\alpha\dot{q}=0$$

It is a second-order differential equation with constant coefficients which can be put in the form:

$$\ddot{q}+2\delta\dot{q}+\omega_0^2q=0$$

where δ is a positive coefficient, called the damping factor (or coefficient) and defined by:

$$\delta=\frac{\alpha}{2a_0}$$

ω_0 is the natural pulsation defined by

$$\omega_0=\sqrt{\frac{b_0}{a_0}}$$

2. Solving the differential equation:

The solution to the differential equation depends on the value of δ compared to ω_0 :

- If $\delta>\omega_0$, the system is said to be overdamped or aperiodic.

– If $\delta=\omega_0$, we say that we have critical damping.

– If $\delta < \omega_0$, we say that the system is pseudoperiodic.

Case where the system is overdamped ($\delta > \omega_0$)

The solution to the differential equation is written in this case:

$$q(t) = A_1 e^{\left[-\delta - \sqrt{\delta^2 - \omega_0^2}\right]t} + A_2 e^{\left[-\delta + \sqrt{\delta^2 - \omega_0^2}\right]t}$$

A 1 and A 2 are constants of integration defined by the initial conditions. The figure below represents q as a function of time in the special case where $q(0) = q_0$ And $\dot{q}(0) = 0$. $Q(t)$ is a function that tends exponentially (without oscillation) towards zero

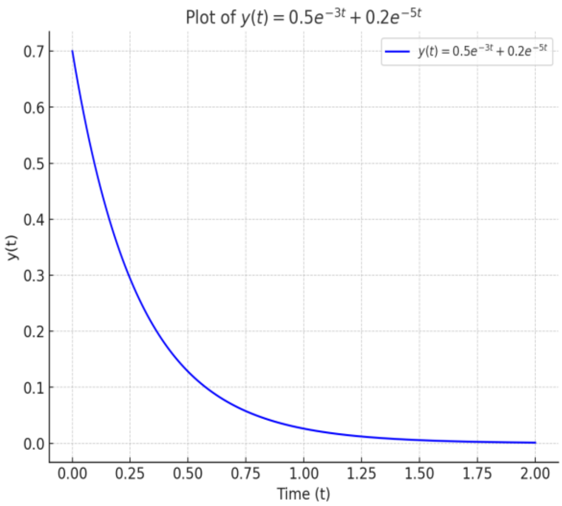


Fig.4. Overdamped regime: variation of q as a function of time

Case of critical damping ($\delta = \omega_0$**):**

The general solution of the differential equation is of the form:

$$q(t) = (A_1 + A_2\, t)e^{-\delta t}$$

In the particular case where $q(0) = q_0$ And $\dot{q}(0) = 0$.

$$q(t) = q_0(1 + \delta\, t)\, e^{-\delta t}$$

$q(t)$ is still a function that tends to zero without oscillation as time increases.

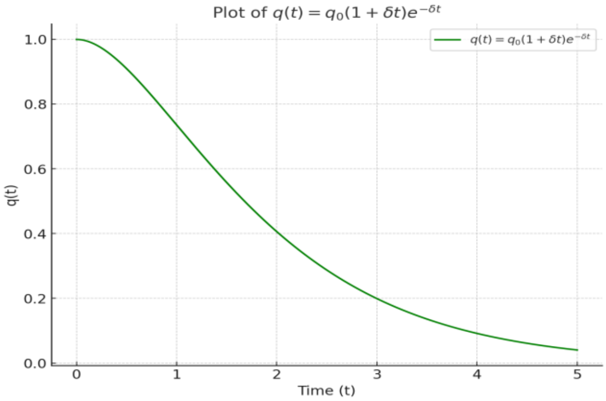


Fig.5. Critically-damped: variation of q as a function of time

Case where the system is underdamped ($\delta < \omega_0$):

The general solution of the differential equation is of the form:

$$q(t) = A\, e^{-\delta t}\, cos\ \ (\omega_A t + \varphi)$$

Or $\omega_A = \sqrt{\omega_0^2 - \delta^2}$ A and φ are two constants of integration determined from the initial conditions. In the particular case where $q(0) = q_0$ and $\dot{q}(0) = 0$, we obtain:

$$A = \frac{\omega_0}{\omega_A} q_0$$

$$\varphi = -\arctan\left(\frac{\delta}{\omega_0}\right)$$

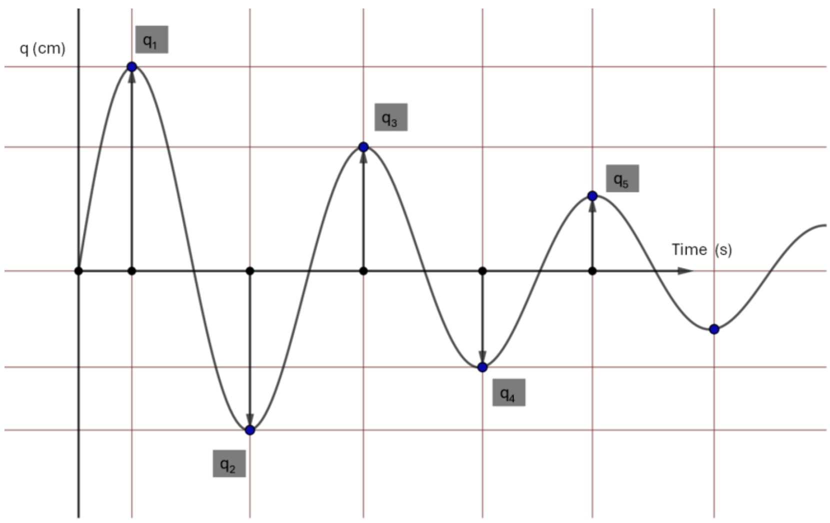


Fig.5. Weakly damped system: variation of q as a function of time

2-3-3. Logarithmic decrement:

In the study of oscillatory systems, particularly free oscillators subjected to weak frictional forces, the concept of logarithmic decrement becomes an essential tool for characterizing the damping behavior of these systems. Logarithmic decrement quantifies the rate at which the amplitude of oscillations decreases over time due to damping.

Definition of Logarithmic Decrement

The logarithmic decrement (δ) is defined as the natural logarithm of the ratio of two successive amplitudes of the oscillation. Mathematically, it can be expressed as:

$$\delta = ln\left(\frac{q(t)}{q(t+T)}\right)$$

Where: q(t) is the amplitude at time t and q(t+T) is the amplitude at the subsequent oscillation cycle. This ratio reflects how much the amplitude decreases from one cycle to the next due to the damping effect.

Physical Interpretation

In the context of a free oscillator with weak friction, the system experiences a gradual loss of energy with each oscillation cycle, leading to a reduction in amplitude. The logarithmic decrement provides a convenient way to quantify this damping effect. A larger logarithmic decrement indicates more significant energy loss and faster decay of amplitude, while a smaller decrement suggests lighter damping and a more prolonged oscillation.

2-3-4. Quality coefficient:

The **Quality Factor** (or **Quality Coefficient**, denoted as Q) is a dimensionless parameter that describes the damping of an oscillator, particularly in systems that exhibit oscillatory motion. It provides a measure of how underdamped an oscillator is and relates to the sharpness of resonance in the system.

Definition

The Quality Factor Q is defined as the ratio of the energy stored in the system to the energy dissipated per cycle of oscillation. In other words, it quantifies how many oscillations a system can perform before the amplitude decreases significantly due to energy loss. Mathematically, it can be expressed as:

$$\mathrm{Q} = 2\pi \times \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}}$$

Alternatively, for a system undergoing harmonic oscillation, the Quality Factor is related to the natural frequency and the damping coefficient and can be expressed as:

$$Q = \frac{\omega_0}{B} = \frac{\omega_0}{2\delta}$$

Physical Interpretation

- **High-Q Oscillators**: If the Q-factor is large, the system is **lightly damped** and oscillates for many cycles before its amplitude diminishes significantly. These systems exhibit sharp resonance peaks and are typically efficient at storing energy.
- **Low-Q Oscillators**: If Q is small, the system is **heavily damped**, meaning the oscillations die out quickly due to energy loss. These systems have broad resonance peaks and lose energy rapidly.

Quality Factor and Resonance

The Quality Factor plays a crucial role in resonance phenomena. A higher Q-factor corresponds to a more selective system that exhibits stronger and sharper resonances. For example, in a high-Q system, a small range of external forcing frequencies will generate a large amplitude of oscillation, while frequencies outside this range will have minimal effect.

Applications

- **Mechanical Systems**: In mechanical oscillators (e.g., springs and masses), Q determines how long a system can oscillate before coming to rest. A high-Q mechanical oscillator retains its motion for longer durations.

- **Electrical Circuits**: In circuits containing inductors and capacitors, Q measures how well the circuit resonates at a given frequency. A high-Q circuit is efficient in filtering signals at a specific frequency.
- **Optical Cavities**: In lasers and optical systems, the Q-factor of an optical cavity determines how well it stores light, with high-Q cavities providing better performance.

Test your comprhension

Problem 1: **Damped Free Oscillations**

Consider a damped system with parameters m=1 kg, k=25 N/m, and b=3 Ns/m. The initial displacement is x_0=0.1 m, and the initial velocity is v_0=0 m/s.

- (a) Derive the equation of motion for the system.
- (b) Solve for x(t).
- (c) Find the damping ratio ζ and classify the type of motion.
- (d) Plot x(t) and discuss how the damping affects the oscillations.

Problem 2:

- **Simple Pendulum Derivation**

- Derive the equation of motion for a simple pendulum (mass m, length l) using the Lagrangian method. Assume small oscillations and verify the linearized solution.

- **Mass-Spring System**

- A block of mass mmm is attached to a spring of stiffness k and moves on a frictionless surface. Use the Lagrangian method to derive the equation of motion. Show that the system undergoes simple harmonic motion and find its angular frequency.

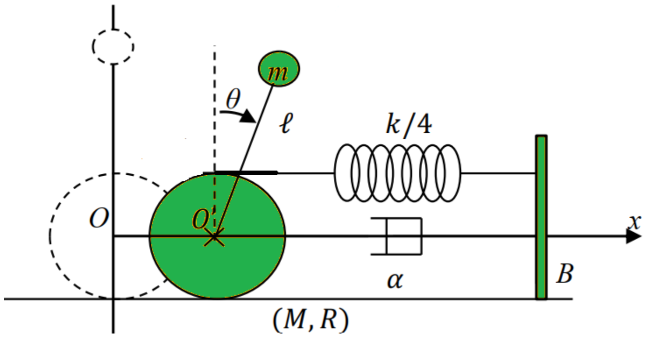
- **Rotational Oscillator**

- A rigid rod of length 2l and mass m pivots at one end. Derive the equation of motion for small angular displacements using the Lagrangian method. What is the natural frequency of oscillation?

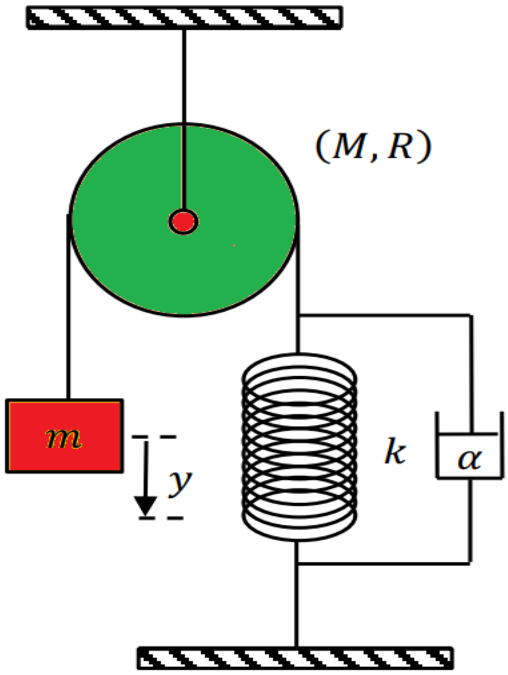
Problem 3:

For Small Amplitudes of the Given Systems, Determine the Following:

1. The number of degrees of freedom of the system.
2. The kinetic energy, potential energy, and Lagrangian of the system.
3. Derive the differential equation of motion and calculate the system's natural frequency.



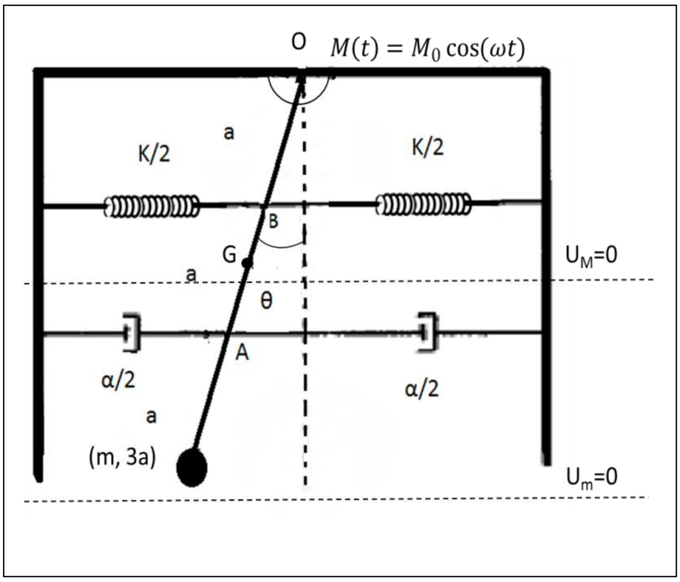
System 1



System 2

Problem 4:

The mechanical system shown in the figure opposite consists of a mass **m** fixed to the end of a rod of mass **M** and length **3a** . This can oscillate without friction, in a vertical plane, around a fixed axis perpendicular to the plane of movement in " **O** ".



Two dampers with a viscous friction coefficient of **α/2** connect point **A (OA=2a)** of the rod. Two identical springs with a stiffness constant of **K/2** , placed horizontally, connect point **B (OB=a)** of the rod. The position of the mass will be identified by the nail θ that the rod makes with the vertical. At equilibrium, the rod is in a vertical position and the two springs are at rest. Only low amplitude movements will be considered

1- Calculate the number of degrees of freedom of the system.
2- Determine the differential equation of motion of the system.
3- When the rod is moved away from its equilibrium position by a fingernail $\boldsymbol{\theta_0 = \pi/60}$, then released without initial speed, it takes on a damped oscillatory motion of pseudo-period $\boldsymbol{T_a = 0.11s}$. It is noted that after $\boldsymbol{40T_a}$ the elongation of the oscillations reaches **80%** of the initial elongation. Calculate the value of the damping coefficient and deduce the values of **α** and **K** . Knowing that M = 2m, m = 111g, a = 10cm and g = 10m/S ^{2} .
4- A moment is applied to the rod $M(t) = M_0 \cos(\omega t)$.
   a) Write the differential equation of motion of the system.
   b) Deduce the steady-state solution.
   c) Give the resonance pulsation and deduce the amplitude at this pulsation.

Solution:

Problem3: System 1

Number of degrees of freedom of the system:

S=3N-n, N=2 and $n = \begin{cases} Z_m = 0 \\ Z_M = 0 \\ x_M = cte \\ f(x_m, y_m) = 0 \\ f(x_m, y_m, y_M) = 0 \end{cases} \Longrightarrow n = 5 \Rightarrow S = 1$

Kinetic energy, potential energy, and Lagrangian of the system:

$$U = \frac{1}{2}(kR^2 - mgl)\theta^2$$

$$T = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(R + l)^2\right]\dot{\theta}^2$$

Dissipation function

$$D = \frac{1}{2}\alpha(R\dot{\theta})^2$$

Lagrangian of the system

L=T-U

$$L = \frac{1}{2}\left[\frac{3}{2}MR^2 + m(R + l)^2\right]\dot{\theta}^2 - \frac{1}{2}(kR^2 - mgl)\theta^2$$

Differential equation of motion and the system's natural frequency

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = 0$$

$$\left[\frac{3}{2}MR^2 + m(R + l)^2\right]\ddot{\theta} + \alpha(R)^2\dot{\theta} + (kR^2 - mgl)\theta = 0$$

Natural frequency:

$$w_0 = \sqrt{\frac{(kR^2 - mgl)}{\left[\frac{3}{2}MR^2 + m(R + l)^2\right]}}$$

System 2:

Number of degrees of freedom of the system:

S=3N-n, N=2 and $n = \begin{cases} Z_m = 0 \\ Z_M = 0 \\ x_M = cte \\ y_M = cte \\ x_m = cte \end{cases} \Longrightarrow n = 5 \Rightarrow S = 1$

Kinetic energy, potential energy, and Lagrangian of the system:

$$U = \frac{1}{2}k(R\theta)^2$$

$$T = \frac{1}{2}(m + \frac{M}{2})R^2\dot{\theta}^2$$

$$L = \frac{1}{2}(m + \frac{M}{2})R^2\dot{\theta}^2 - \frac{1}{2}k(R\theta)^2$$

Dissipation function

$$D = \frac{1}{2}\alpha(R\dot{\theta})^2$$

Differential equation of motion and the system's natural frequency

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - (\frac{\partial L}{\partial \theta}) + (\frac{\partial D}{\partial \dot{\theta}}) = 0$$

$$\left(m + \frac{M}{2}\right)R^2\ddot{\theta} + \alpha(R)^2\dot{\theta} + k(R)^2\theta = 0$$

Natural frequency:

$$w_0 = \sqrt{\frac{k}{\left[m + \frac{M}{2}\right]}}$$

Problem 4:

$\boldsymbol{S = 3N - n \; on \; a \; N = 3, n} \begin{cases} \boldsymbol{Z_1 = 0} \\ \boldsymbol{Z_2 = 0} \\ \boldsymbol{f(X_1, Y_1) = 0} \\ \boldsymbol{f(X_2, Y_2) = 0} \\ \boldsymbol{f(X_1, Y_1, X_2, Y_2) = 0} \end{cases} \Rightarrow \boldsymbol{n = 5 \; donc \; S = 1}$one degree of freedom; a generalized coordinate θ.

Potentilla energy

U= mg $\frac{3a}{2}(1-\cos\theta)$+ Mg $\frac{a}{2}(1-\cos\theta)$+$\frac{1}{2}\left(\frac{K}{2}\right)(\text{a}\sin\theta)^2 + \frac{1}{2}\left(\frac{K}{2}\right)(a\,sin\,\theta)^2$

We have M=m, for low amplitudes.

U = $mga\theta^2 + \frac{1}{2}Ka^2\theta^2$

Kinetic energy

T=$\frac{1}{2}m(3a)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$où $I = \frac{1}{3}m(3a)^2 \Rightarrow T = 6ma^2\dot{\theta}^2$

L= TU= $6ma^2\dot{\theta}^2$-$(mga + \frac{1}{2}Ka^2)\theta^2$

The dissipated energy D=$\frac{1}{2}\left(\frac{\alpha}{2}\right)(2a\dot{\theta})^2 + \frac{1}{2}\left(\frac{\alpha}{2}\right)(2a\dot{\theta})^2 = 2\alpha a^2\dot{\theta}^2$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \frac{\partial D}{\partial \dot{\theta}} = 0 \Rightarrow 12ma^2\ddot{\theta} + 2\left(mga + \frac{1}{2}Ka^2\right)\theta + 4\alpha a^2\dot{\theta} = 0$$

$\Rightarrow \ddot{\theta} + \dot{\theta} + \left(\frac{g}{6a} + \frac{K}{12m}\right)\theta$=0

We have: $2\delta = \frac{\alpha}{3m}$, $\omega_0^2 = \left(\frac{g}{6a} + \frac{K}{12m}\right)$, T $_a$ =0.11S,$\frac{\theta(t+40T_a)}{\theta(t)} = 0{,}8 \Rightarrow \ln(0{,}8) = -40\delta T_a = $ $> \delta = 5.10^{-2}S^{-1}$

$$\omega_D^2 = \omega_0^2 - \delta^2 et\, \omega_D = \frac{2\pi}{T_a} \Rightarrow \omega_0^2 = \omega_D^2 + \delta^2 = \left(\frac{g}{6a} + \frac{K}{12m}\right) \Rightarrow K$$
$$= 12m\left[\omega_D^2 + \delta^2 - \frac{g}{6a}\right] \Rightarrow$$

$$K = 54{,}23Nm^{-1}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \frac{\partial D}{\partial \dot{\theta}} = M_0 \cos(\omega t) => \ddot{\theta} + \dot{\theta} + \left(\frac{g}{6a} + \frac{K}{12m}\right)\theta = \frac{M_0\, cos(\omega t)}{12ma^2}$$

In steady state

$$\theta_p(t) = A\cos(\omega t + \varphi)\ où\ A = \frac{\frac{M_0}{12ma^2}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}\ et\ \varphi = -arctg\ \frac{2\delta\omega}{\omega_0^2 - \omega^2}$$

The resonance pulse$\omega = \omega_0$

$$A = \frac{M_0}{24\delta\omega_0 ma^2}$$

Chapter 3: Forced Oscillations of One-degree-of-freedom Systems

4-1. Introduction:

In the study of physical chemistry, the principles of oscillatory motion are essential for understanding a wide range of phenomena, from molecular vibrations to macroscopic mechanical systems. Chapter 3 focuses on the topic of forced oscillations in one-degree-of-freedom systems, where external periodic forces influence the motion of these systems.

Forced oscillations are crucial in various chemical and physical applications, including spectroscopy techniques, where the interaction between light and matter can induce oscillatory behavior in molecular systems. Understanding how systems respond to external forces is vital for predicting their behavior in different environments, particularly when resonance occurs, which can lead to significant amplitude increases and altered reaction rates.

This chapter will begin with the basic concepts of forced oscillations, establishing the mathematical framework necessary to describe the dynamics of one-degree-of-freedom systems under external influences. We will examine the response of these systems to harmonic forces, focusing on both the steady-state and transient responses. The effects of damping, which plays a critical role in determining the amplitude and phase of oscillations, will also be discussed.

Real-world applications in chemistry, such as vibrational spectroscopy and molecular interactions, will be highlighted to illustrate the relevance of these concepts. By the end of this chapter, students will have a solid understanding of forced oscillations, empowering them to analyze and predict the behavior of systems influenced by external periodic forces.

1.4 Case of a time-dependent external force

In the more general case where a time-dependent external force acts on a system, alongside frictional forces derived from a dissipation function D, let F_{eq} denote the q-component of the external force or Feq is the generalized component of the external non-conservative force. Under these conditions, Lagrange's equation can be formulated in either of the following two equivalent forms:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = F_{eq} - \beta \dot{q} \qquad (3.1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) = F_{eq} \qquad (3.2)$$

Eq (3.2) called the generalized Lagrange equation in the presence of non-conservative forces.

3-2. Differential equation of the mass-spring-damper system in forced oscillation:

3.1 Differential equation:

The general form of the Lagrange equation for systems with one degree of freedom

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{qext} \qquad (3.3)$$

Or F_{qext} is the generalized force associated with F_{ext} and where the dissipation function is $D = \frac{1}{2}\alpha \dot{q}^2$.

For small amplitude oscillations, the Lagrange function could be put into a quadratic form of q et $\dot{q}$

$$L = T - U = \frac{1}{2}a_0\dot{q}^2 - \frac{1}{2}b_0 q^2 \qquad (3.4)$$

Hence the differential equation of motion

$$a_0\,\ddot{q} + \alpha\,\dot{q} + b_0\,q = F_{qext} \qquad (3.5)$$

This equation can be put in the form of a second-order differential equation with constant coefficients, with second member.

$$\ddot{q} + 2\delta\,\dot{q} + w_0^2\,q = A(t) \qquad (3.6)$$

Where: $2\delta = \frac{\alpha}{a_0}, w_0^2 = \frac{b_0}{a_0}$, and $A(t) = \frac{F_{qext}}{a_0}$

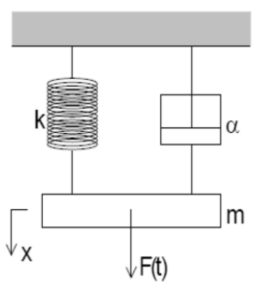


Fig.1: Mass-spring-shock absorber system

3.2 Mass-spring-shock absorber system:

Consider the mechanical example in the figure opposite.

Subjected to an external force $\vec{F}$ applied to the mass m .

Let us calculate F_x the generalized conjugate force.

We have the position $\vec{r} = x\,\vec{i}$ vector and $\vec{F} = F\,\vec{i}$ therefore $F_x = \vec{F}\cdot\frac{\partial \vec{r}}{\partial x} = F_{qext}$

The differential equation of motion is: $\ddot{x} + 2\delta\,\dot{x} + \omega_0^2\,x = A(t)$ …………… .(*3.7*)

$$\text{with } 2\delta = \frac{\alpha}{2m}$$

$$\omega_0^2 = \frac{k}{m} \qquad \text{and} \qquad A(t) = \frac{F_{qext}}{m}$$

3.3. Solution of the differential equation:

The solution to the equation:

$$\ddot{x} + 2\delta\,\dot{x} + \omega_0^2\,x = \frac{F_{qext}}{m} \quad \text{is} \quad x_G(t) = x_H(t) + x_P(t) \quad (3.8)$$

Where $x_G(t)$: is the general solution, $x_H(t)$: is the homogeneous solution and $x_P(t)$: is the particular solution which is related to the second member of the equation; the homogeneous solution vanishes for some time and only the particular solution remains, so the general solution of the equation will be the particular solution $\left[x_G(t) \approx x_P(t)\right]$.

The time interval during which the homogeneous solution is non-negligible is called the transient regime. At the end of this transient regime begins the time interval for which the homogeneous solution is quasi-zero and for which the solution $x_G(t) \approx x_P(t)$; this regime is called the permanent or stationary regime.

3.3.1 Special case where $A(t) = A_0 \cos(\Omega t)$:

The particular solution $x_P(t)$ is of the same form as the right-hand side, so we are looking $x_P(t)$ for the form $x_P(t) = A\cos(\Omega t + \varphi)$.

We have:

$$\dot{x}_P(t) = -\mathrm{A}\,\Omega\sin(\Omega\,\mathrm{t}+\varphi)$$
$$\ddot{x}_P(t) = -\mathrm{A}\,\Omega^2\cos(\Omega\,\mathrm{t}+\varphi)$$

Substituting into the equation (T) we will have .

$$-\mathrm{A}\,\Omega^2\cos(\Omega\,\mathrm{t}+\varphi) - 2\delta\,\mathrm{A}\,\Omega\sin(\Omega\,\mathrm{t}+\varphi) + \omega_0^2\,\mathrm{A}\cos(\Omega\,\mathrm{t}+\varphi) = \mathrm{A}_0\cos(\Omega\,\mathrm{t}) \quad (3.9)$$

We pose

$$\mathrm{A}_0\cos(\Omega\,\mathrm{t}) = \mathrm{A}_0\cos(\Omega\,\mathrm{t}+\varphi-\varphi) \qquad (3.10)$$

We will have

$$\mathrm{A}_0\cos(\Omega\,\mathrm{t}) = \mathrm{A}_0\left\{\cos(\Omega\,\mathrm{t}+\varphi)\cos(\varphi) + \sin(\Omega\,\mathrm{t}+\varphi)\sin(\varphi)\right\} \qquad (3.11)$$

From where

$$(T) \Rightarrow \left(\omega_0^2 A - A\Omega^2\right)\cos(\Omega t+\varphi) - 2\delta A\Omega\sin(\Omega t+\varphi) = \mathrm{A}_0\cos(\varphi)\cos(\Omega\,\mathrm{t}+\varphi) + \mathrm{A}_0\sin(\varphi)\sin(\Omega\,\mathrm{t}+\varphi)$$

$$\Rightarrow \begin{cases} \omega_0^2 A - A\Omega^2 = \mathrm{A}_0\cos(\varphi) \\ -2\delta A\Omega = \mathrm{A}_0\sin(\varphi) \end{cases} \Rightarrow \begin{cases} \tan(\varphi) = \dfrac{2\delta\,\Omega}{\Omega^2-\omega_0^2} \\ A_0^2\left(\cos^2(\varphi)+\sin^2(\varphi)\right) = A^2\left[4\delta^2\Omega^2 + (\omega_0^2-\Omega^2)^2\right] \end{cases}$$

$$\Rightarrow \begin{cases} \varphi = \arctan\left(\dfrac{2\delta\,\Omega}{\Omega^2-\omega_0^2}\right) \\ A = \dfrac{A_0}{\sqrt{4\delta^2\Omega^2 + (\omega_0^2-\Omega^2)^2}} \end{cases}$$

So the solution is: $x_P(t) = \mathrm{A}\cos(\Omega\,\mathrm{t}+\varphi)$ with $\begin{cases} \varphi = \arctan\left(\dfrac{2\delta\,\Omega}{\Omega^2-\omega_0^2}\right) \\ A = \dfrac{A_0}{\sqrt{4\delta^2\Omega^2 + (\omega_0^2-\Omega^2)^2}} \end{cases}$ (3.12)

3. Study of the variations of the amplitude and phase as a function of the excitation pulsation:

The amplitude ***A*** has a maximum if $\dfrac{dA}{d\Omega} = 0$

We have:

$$\frac{dA}{d\Omega} = 0 \Rightarrow \frac{A\left\{8\delta^2 - 4\omega_0^2 + 4\Omega^2\right\}}{\left[4\delta^2\Omega^2 + (\omega_0^2 - \Omega^2)^2\right]^{3/2}} = 0 \Rightarrow \Omega_R = \sqrt{\omega_0^2 - 2\delta^2} \quad (3.13)$$

There is a maximum at the pulsation $\Omega_R = \sqrt{\omega_0^2 - 2\delta^2}$ only if the damping is sufficiently low so that $\delta < \frac{\omega_0}{\sqrt{2}}$. At this pulsation, called the resonance pulsation, the system is said to enter resonance and the amplitude ***A*** is maximum; it is worth

$$A = \frac{A_0}{2\delta\sqrt{\omega_0^2 - \delta^2}} \quad (3.14)$$

The figure representing the variations of *A* as a function of the excitation pulsation Ω is called the amplitude resonance curve. It is noted that at the pulsation ω_0 , the phase shift φ is equal to $-\dfrac{\pi}{2}$, and that at resonance $\varphi = -\arctan\left[\dfrac{\sqrt{\omega_0^2 - 2\delta^2}}{\delta}\right]$ (3.15)

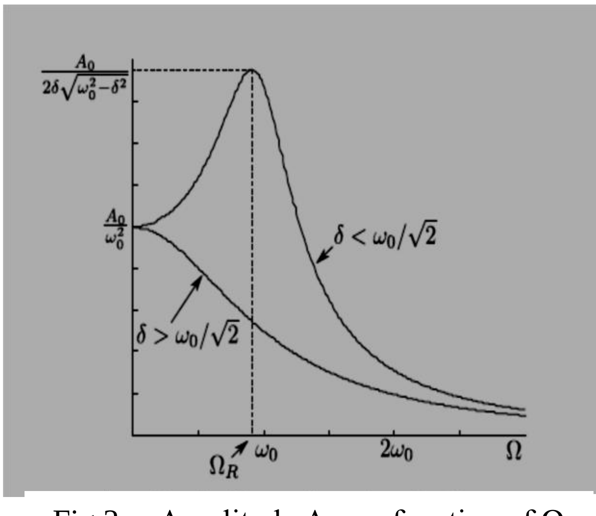


Fig.2.a: Amplitude A as a function of Ω

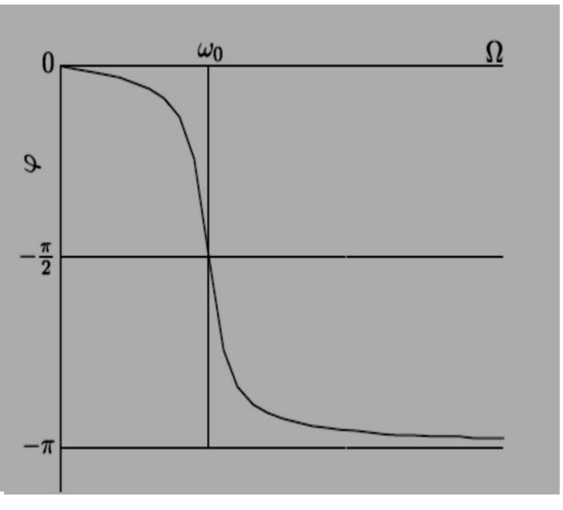


Fig.2.b: Phase shift φ as a function of Ω

For low damping $\delta << \omega_0$, the resonance frequency is very little different from the natural pulsation, $\Omega \approx \omega_0$ In this case, the vibration amplitude at resonance $\boldsymbol{A}_{max}$ is equal to:

$$\frac{A_0}{2\delta\omega_0} \qquad (3.16)$$

Behavior of Amplitude and Phase

- **Amplitude** A(ω)A:
  - At **low frequencies** ($\omega \ll \omega_0$): The amplitude increases as the frequency increases.
  - At **resonance** ($\omega = \omega_0$): The amplitude reaches a maximum, provided the damping is small.
  - At **high frequencies** ($\omega \gg \omega_0$): The amplitude decreases, and the system can no longer keep up with the fast-driving force.
- **Phase** ϕ(ω):
  - At **low frequencies** ($\omega \ll \omega_0$): The phase shift is near zero, meaning the displacement is almost in phase with the driving force.
  - At **resonance** ($\omega = \omega_0$): The phase shift is π/2, meaning the displacement lags the driving force by a quarter cycle.
  - At **high frequencies** ($\omega \gg \omega_0$): The phase shift approaches π\piπ, meaning the displacement is out of phase with the driving force.

b. Speed study

The velocity v(t) is the time derivative of the displacement x(t):

$$v(t) = \frac{dx}{dt} \qquad (3.17)$$

For x(t)=Acos(ωt−ϕ), the velocity is:

$$v(t) = -A\omega\sin(\omega t - \phi) \qquad (3.18)$$

The amplitude of the velocity is then:

$$V = A\omega \qquad (3.19)$$

To express V in terms of the parameters of the system, we first recall that the amplitude of the displacement A is given by:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+(2\delta\omega)^2}} \qquad (3.20)$$

Now, the velocity amplitude V is:

$$V = \frac{F_0\omega/m}{\sqrt{(\omega_0^2-\omega^2)^2+(2\delta\omega)^2}} \qquad (3.21)$$

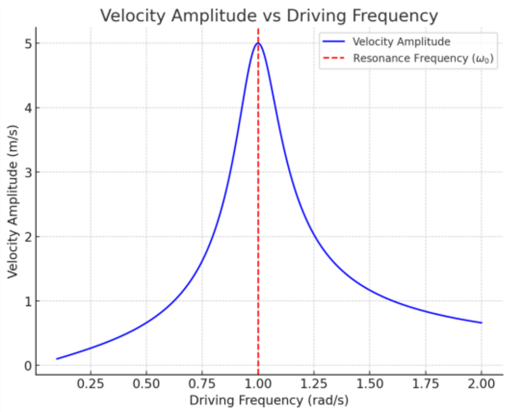


Fig.3.a:Velocity resonance curve

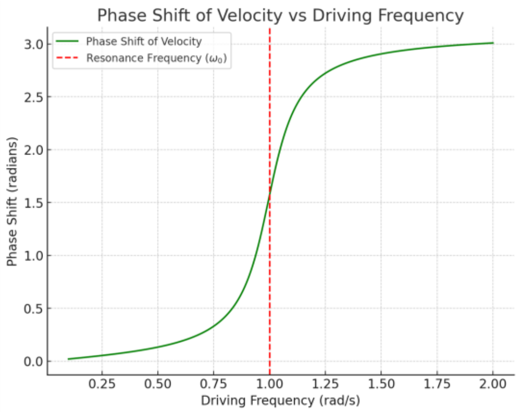


Fig.3.b:Phase shift φ of the velocity as a function of Ω

Behavior of Velocity as a Function of Driving Frequency (see Fig.3.a)

The amplitude of the velocity V depends on the driving frequency ω in the following way:

- At low frequencies ($\omega \ll \omega_0$): The term $(\omega_0^2 - \omega^2)$ dominates, so Vis small because the system is slow to respond to the external force.
- At resonance $(\omega_0^2 \approx \omega^2)$: The velocity amplitude V reaches a maximum because the driving frequency matches the natural frequency of the system. This is when the system oscillates with maximum energy, and the velocity becomes largest.
- t high frequencies ($\omega \gg \omega_0$): The damping term $(2\delta\omega)$ and the ω^2 terms dominate, and the velocity amplitude V decreases, because the system can no longer respond quickly enough to the high-frequency driving force.

Behavior of the Phase Shift (see Fig.3.a)

- At **low frequencies** ($\omega \ll \omega_0$): The velocity is almost in phase with the driving force, so the phase shift ϕ is close to 0.

- At **resonance** (ω=ω_0): The phase shift is π/2 (90°), meaning the velocity lags the driving force by a quarter cycle.
- At **high frequencies** (ω≫ω_0): The phase shift approaches π (180°), meaning the velocity is almost completely out of phase with the driving force (the system moves in the opposite direction of the force).

Energy balance:

$P_F(t)$ be the instantaneous power supplied by the external force $F(t)$ to the system. In steady state, we obtain:

$$P_F(t) = F(t)\dot{x}(t) = F_0 A\Omega\, cos(\Omega t)\ sin(\Omega t + \varphi) \quad (3.21)$$

Let $< P_F >$ *be* the average value over a period of $P_F(t)$:

$$\langle P_F \rangle = \frac{1}{T}\int_0^T P_F(t)dt \quad \Rightarrow \langle P_F \rangle = -\frac{\Omega}{2}F_0 A\, sin(\varphi) = \frac{1}{2}\alpha\, \Omega^2 A^2 \quad (3.22)$$

car $sin(\varphi) = \frac{-2\delta A\Omega}{A_0}$ and $\alpha = 2m\delta$ (3.23)

Let us compare this value with the average value $< P_D >$ of the power dissipated by the viscosity friction forces. The instantaneous value of this dissipated power is written as:

$$P_D(t) = \alpha\dot{x}(t)\dot{x}(t) = \alpha(A\Omega)^2\ sin^2(\Omega t + \varphi) \quad (3.24)$$

$$\langle P_D \rangle = \frac{1}{T}\int_0^T D_F(t)dt \Rightarrow \langle D_F \rangle = \frac{1}{2}\alpha\, \Omega^2 A^2 \quad (3.24)$$

The study of the variations of the mean value of the power $< P > = < P_F > = < P_D >$ as a function of the excitation pulsation shows that the maximum value of the mean power is obtained for $\Omega = \omega_0$ whatever the value of δ The maximum value of the mean power dissipated or supplied is in this case: $\langle P \rangle_{Max} = \frac{F_0^2}{2\alpha}$ (3.25)

The figure below represents the variations, as a function of , of the average power dissipated by the friction forces (or equivalently the average power provided by the external force).

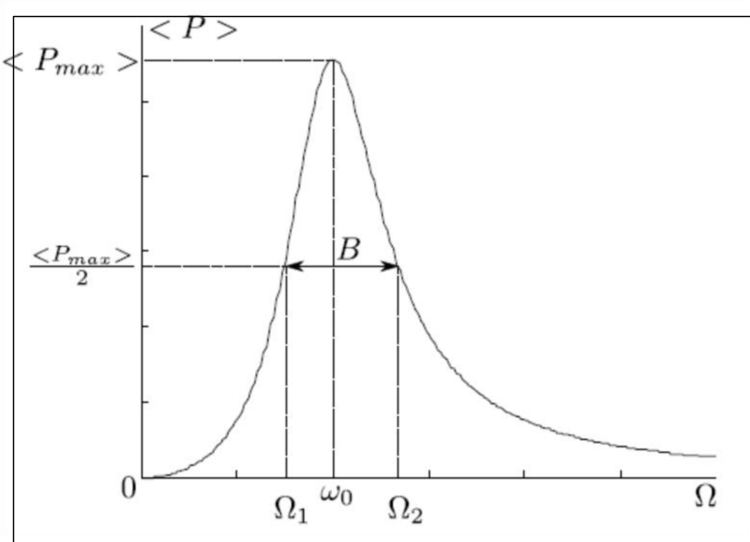


Fig.4. Resonance curve for power as function of frequencies

Bandwidth

We define bandwidth as the band of pulsations around $\Omega = \omega_0$ which $\langle P \rangle \geq \frac{1}{2}\langle P \rangle_{Max}$.

The two pulsations Ω_1 et Ω_2, located on either side of the pulsation ω_0 and for which $\langle P \rangle \geq \frac{1}{2}\langle P \rangle_{Max}$, are called cut-off pulsations. The bandwidth *B* is written:

$$B = \Omega_2 - \Omega_1 \quad (3.26)$$

The calculation of *B* consists of finding the two pulsations for which $\langle P \rangle \geq \frac{1}{2}\langle P \rangle_{Max}$.

We obtain the expression for the bandwidth *B* :

$$B = \Omega_2 - \Omega_1 = \delta \quad (3.27)$$

Quality coefficient of an oscillator

The quality coefficient of an oscillator is defined by the ratio of the natural pulsation ω_0 to the bandwidth *B* :

$$Q = \frac{\omega_0}{B} \quad (3.28)$$

3.4 Mechanical impedance:

3.4.1 Definition:

Consider a mechanical system subjected to a sinusoidal force $F(t) = F_0\ cos\ (t)$. In steady state, the point of application of this force moves with a speed . The ratio of the complex amplitudes of the force $v(t) = V_0\ cos\ (t+\varphi) F$ and the speed v is called the input mechanical impedance of the mechanical system.

$$\underline{Z} = \frac{\underline{F}}{\underline{V}} \text{ Or } \underline{F} = F_0 e^{j.0} = F_0 \quad \text{et} \quad \underline{V} = V_0 e^{j\varphi} \qquad (3.29)$$

We note that $\underline{F}$ is real on the other hand $\underline{V}$ can be imaginary to the real according to the phase shift φ.

3.4.2 Mechanical impedances:

Shock absorber :

In the case of a shock absorber, the applied force is related to the speed by

$$F = \alpha\ v \quad (3.30)$$

We deduce the complex impedance of a shock absorber

$$\underline{Z}_{\alpha} = \alpha \quad (3.31)$$

Mass

In the case of a mass, the fundamental relation of dynamics is written

$$F = m\frac{dv}{dt} \ (3.32)$$

We deduce the complex impedance of a mass

$$\underline{Z}_m = im\Omega = m\ \Omega e^{j\frac{\pi}{2}} \quad (3.33)$$

Spring

In the case of a spring of stiffness k , the applied force f applied to the spring is expressed as a function of the elongation by

$$F = kx \quad (3.34)$$

We deduce the complex impedance of a spring

$$\underline{Z}_k = \frac{k}{i\Omega} = -i\frac{k}{\Omega} = \frac{k}{\Omega}e^{-j\frac{\pi}{2}} \quad (3.35)$$

3.4.3 Power

The average value, over a period, of the power supplied is

$$\langle P_F \rangle = -\frac{\Omega}{2}F_0 A\sin(\varphi) = \frac{1}{2}\mathrm{Re}\left[\underline{Z}_E\right]A^2 \quad (3.36)$$

3.4.4 Applications:

Resonant mechanical system

Consider a mechanical system consisting of a spring of stiffness k , a damper with viscous friction coefficient α and a mass m subjected to a sinusoidal force $F(t) = F_0 \cos(t)$. The input impedance of this system is

$$\underline{Z_E} = \alpha + i(m\,\Omega - \frac{k}{\Omega}) \; (3.37)$$

At the resonance $\left(\Omega = \omega_0 = \sqrt{\frac{k}{m}}\right)$, the impedance module is $\underline{Z}_E = \alpha$ When the pulsation $\Omega \to \infty$, the impedance $\underline{Z}_E \approx im\,\Omega$.

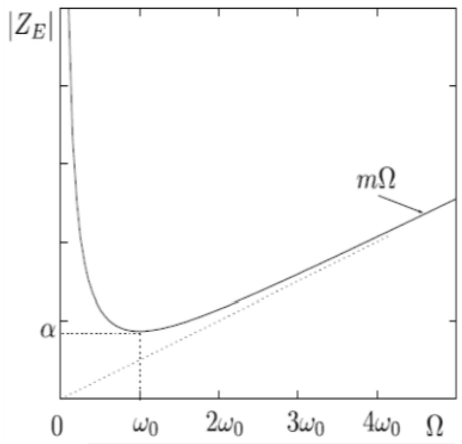


Fig.5.a. Input amplitude module

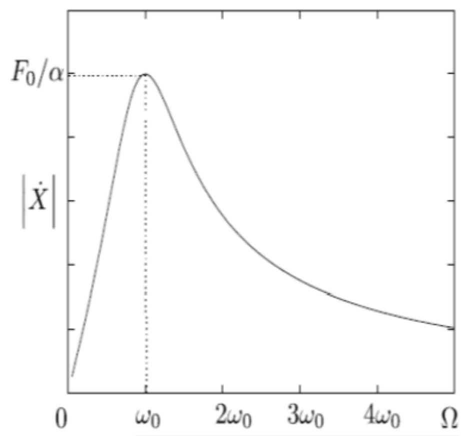


Fig.5.b. Velocity amplitude

the displacement of mass ***m*** and let ***y*** the displacement of the point of application of the force

F (t) . To calculate the input impedance of this system, we must first write the differential equations of motion:

$$\begin{aligned} m\ddot{x} &= k(x-y) \\ F &= k(x-y) \end{aligned} \quad (3.38)$$

Using complex notation, we obtain the input impedance:

$$\underline{Z}_E = \frac{\underline{F}}{\underline{\dot{Y}}} = -i\,\frac{km}{\left[m\,\Omega - \dfrac{k}{\Omega}\right]} \quad (3.39)$$

The antiresonance pulsation is $\omega_0 = \sqrt{\frac{k}{m}}$. When $\Omega = \omega_0$, the velocity $\dot{Y}$ is zero while the impedance modulus is ∞ . When the pulsation $\Omega \to \infty$, the impedance $Z_E \to 0$.

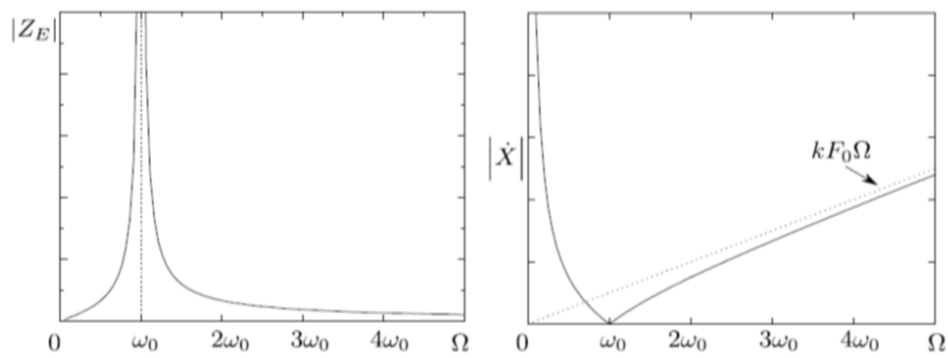


Fig.6.a. Input impedance module   Fig.6.b. Velocity amplitude

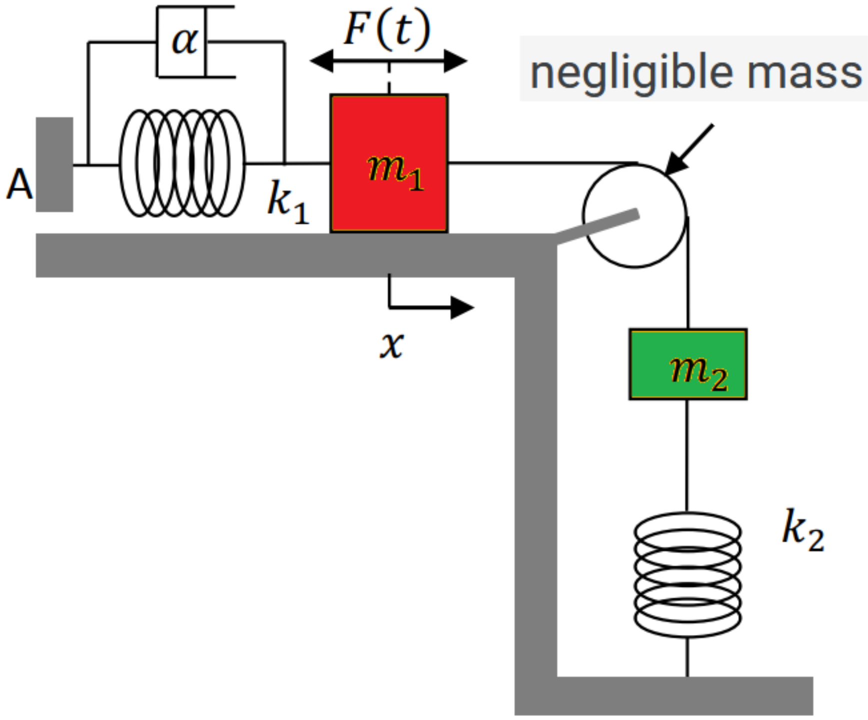
Supplementary Exercises

Problem 1:

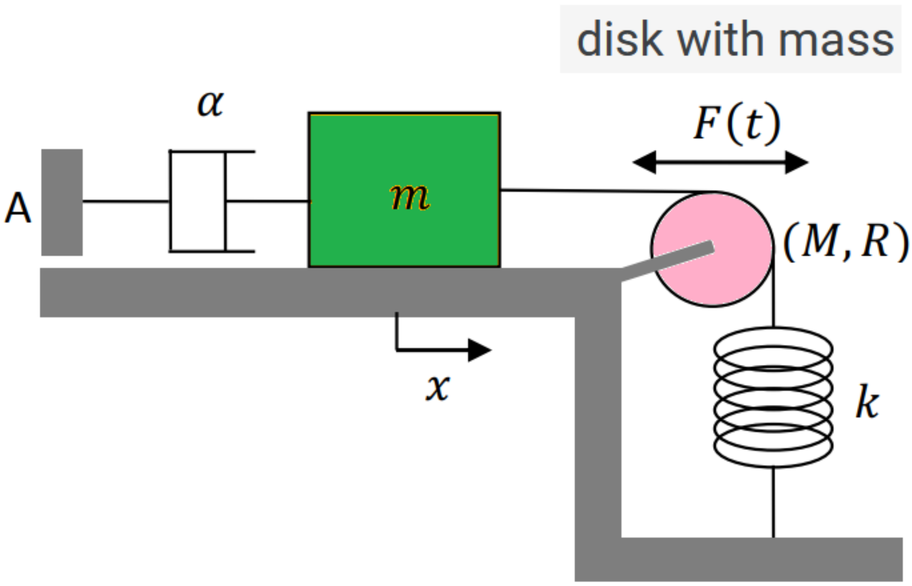
For the given systems, determine the following:

1. The number of degrees of freedom of the system.

2. The kinetic energy, potential energy, and Lagrangian of the system.
3. Derive the differential equation of motion and calculate the system's natural frequency.



System 1



System 2

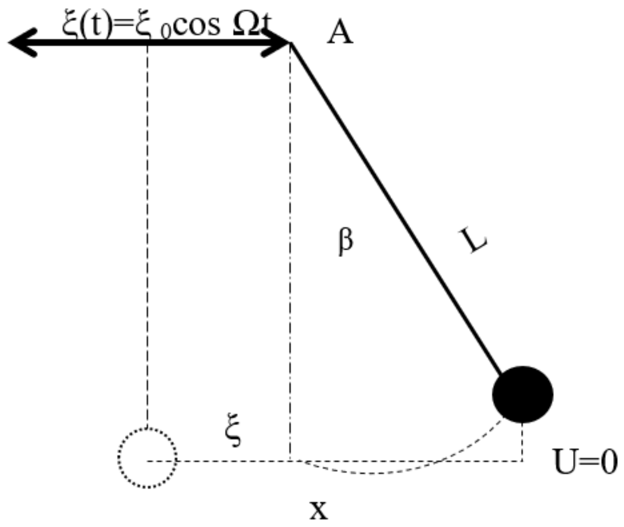
Problem 2:

A simple pendulum of length L and mass M, suspended from a point A, the latter undergoes a displacement $\xi(t)=\xi_0 \cos \Omega t$ [mm].

If the expression of the friction force (air-mass) $\overrightarrow{F_{fr}} = -\alpha \,\vec{\dot{x}}$.

For low amplitudes.

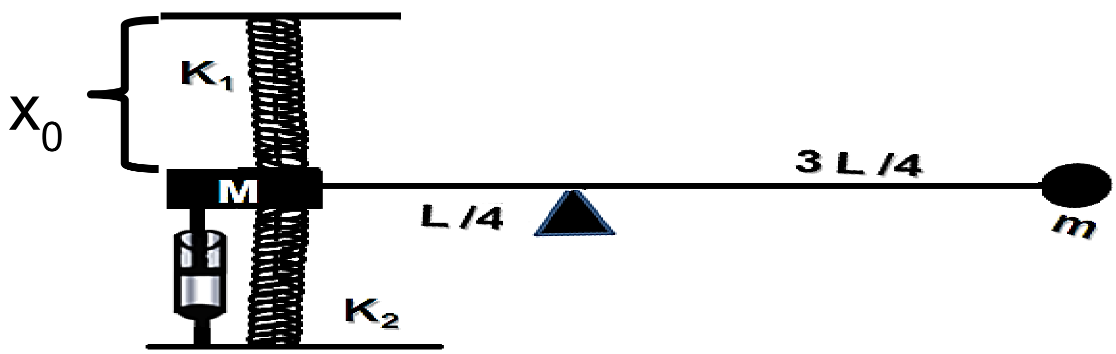
1) What is the nature of the movement?
2) Calculate kinetic energy?
3) Calculate potential energy?
4) Deduce the Lagrangian of the system, and the differential equation of motion?
5) Give the equation of motion if $\delta < \omega_0$?



Problem 3:

M=8kg, m=2kg, $k_1=2x10^3$ N/m, $k_2=k_1/2$, α =200N.s /m, g=10N/s ^{2} , L=1m.

1- Determine the number of degrees of freedom.
   By moving the mass m away from its equilibrium angle γ, by an angle $\theta_0 = 0.15rd$
2- Calculate the angle γ that the bar makes with the horizontal axis for a stable equilibrium?
3- For small amplitudes: Find the kinetic and potential energy, and deduce the Lagrangian of the system.
4- Give its pseudo-period T_a ?
   If the mass m is subjected to a vertical and harmonic force: $\vec{F}(t) = F_0 \cos(9.26t)\vec{j}$
5- Find the differential equation of motion.
6- For the steady state, find the solution to the differential equation of motion.
7- Study the average power provided by the external force.



Solution:

Problem 1:

System 1:

Number of degrees of freedom of the system:

S=3N-n, N=2 and $n = \begin{cases} Z_{m1} = 0 \\ Z_{m2} = 0 \\ y_{m1} = cte \\ x_{m2} = 0 \\ x_{m1} = y_{m2} \end{cases} \Longrightarrow n = 5 \Rightarrow S = 1$

Kinetic energy, potential energy, and Lagrangian of the system:

$$U = \frac{1}{2}(k_1 + K_2)x^2$$

$$T = \frac{1}{2}[m_1 + m_2]\dot{x}^2$$

Dissipation function

$$D = \frac{1}{2}\alpha(\dot{x})^2$$

Lagrangian of the system

L=T-U

$$L = \frac{1}{2}[m_1 + m_2]\dot{x}^2 - \frac{1}{2}(k_1 + K_2)x^2$$

Differential equation of motion and the system's natural frequency

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) + \left(\frac{\partial D}{\partial \dot{x}}\right) = F_G^x$$

$$F_G^x = \overrightarrow{F_{ex}}\frac{\partial \vec{r}}{\partial x} = F_{ex}$$

$$[m_1 + m_2]\ddot{x} + \alpha\dot{x} + (k_1 + K_2)x = F_{ex}$$

Natural frequency:

$$w_0 = \sqrt{\frac{(k_1 + K_2)}{[m_1 + m_2]}}$$

System 2:

Number of degrees of freedom of the system:

S=3N-n, N=2 and $n = \begin{cases} Z_m = 0 \\ Z_M = 0 \\ y_M = cte \\ x_M = cte \\ y_m = cte \end{cases} \Longrightarrow n = 5 \Rightarrow S = 1$

Kinetic energy, potential energy, and Lagrangian of the system:

$$U = \frac{1}{2}kx^2 + U_{eq}$$

$$T = \frac{1}{2}\left[m + \frac{M}{2}\right]\dot{x}^2$$

Dissipation function

$$D = \frac{1}{2}\alpha(\dot{x})^2$$

Lagrangian of the system

L=T-U

$$L = \frac{1}{2}\left[m + \frac{M}{2}\right]\dot{x}^2 - \frac{1}{2}kx^2 + U_{eq}$$

Differential equation of motion and the system's natural frequency

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) + \left(\frac{\partial D}{\partial \dot{x}}\right) = F_G^x$$

$$F_G^x = \overrightarrow{F_{ex}}\frac{\partial \vec{r}}{\partial x} = F_{ex}R$$

$$\left[m + \frac{M}{2}\right]\ddot{x} + \alpha\dot{x} + kx = F_{ex}R$$

Natural frequency:

$$w_0 = \sqrt{\frac{k}{\left[m + \frac{M}{2}\right]}}$$

Chapter 4: Free Oscillations of Two-degree-of-freedom Systems

4-1. Introduction:

In this chapter, we delve into the study of **free oscillations in two-degree-of-freedom systems**, a key concept that extends our understanding of vibrational systems beyond the simpler single-degree-of-freedom cases. For chemistry students, this is particularly important, as molecular structures and interactions often involve multiple degrees of vibrational freedom, and the principles learned here apply directly to the study of molecular vibrations and spectroscopy.

A **two-degree-of-freedom system** refers to a system where two independent variables, or coordinates, are required to fully describe its motion. Unlike simpler systems, these can vibrate in more complex patterns, often displaying interactions between the different parts of the system, leading to phenomena such as **normal modes** and **natural frequencies**. Understanding these behaviors helps explain how molecules oscillate and how energy is distributed in chemical systems.

In this chapter, we will:

1. Derive the equations of motion for two-degree-of-freedom systems.
2. Explore **normal modes**, which describe how the system naturally vibrates.
3. Identify the **natural frequencies** of the system, which are key to understanding resonance and energy behavior.
4. Apply this knowledge to practical examples in molecular vibrations and spectroscopy, which you will encounter more deeply in later studies.

By mastering these concepts, you'll gain a solid foundation that will be critical when you analyze more complex molecular systems and interpret vibrational spectroscopy data in your future chemistry courses.

4-2. **Mass-spring system in translation:**

Systems that require two independent coordinates q_1 , q_2 to specify their positions are called two-degree-of-freedom systems.

Examples:

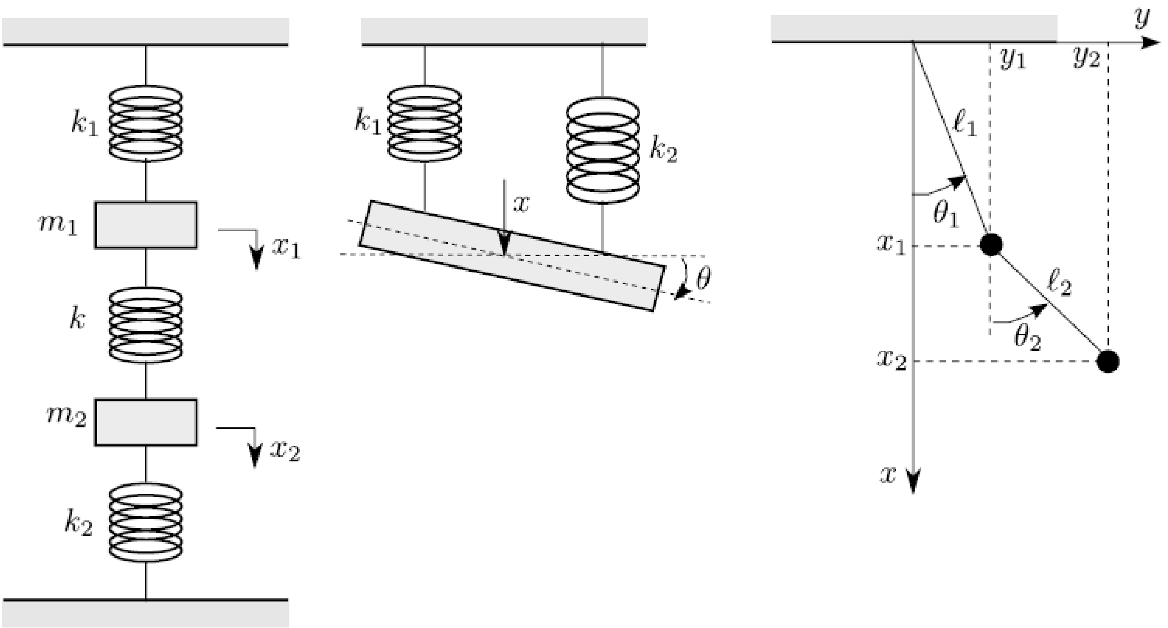


Fig.1: Systems with two degrees of freedom

For the study of systems with two degrees of freedom, it is necessary to write two differential equations of motion which can be obtained from Lagrange's equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = 0 \end{cases} \quad (4.1)$$

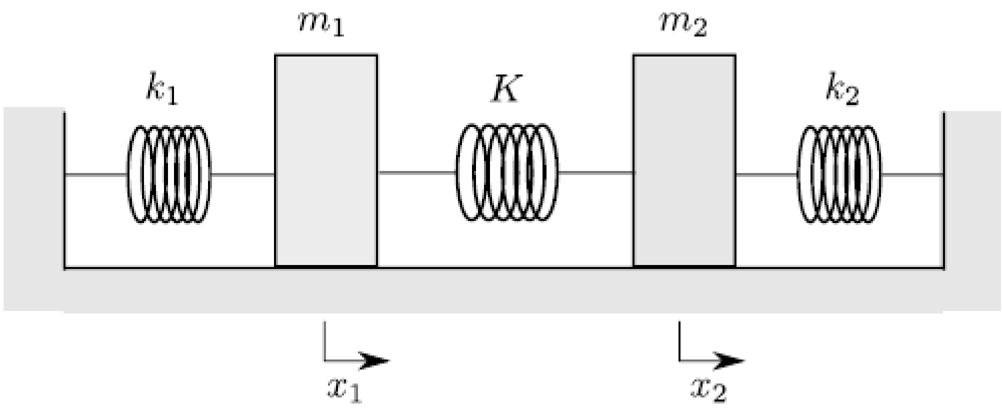


Fig.2: Mass-spring system in translation

4-2-1. **Differential equations of motion:**

Kinetic and potential energies of the system:

$$T = T_1 + T_2$$

$$U = U_e + U_g$$

$$U_e = U_{e1} + U_{e2} + U_{e3} \text{ et } U_g = 0$$

$$T_1 = \tfrac{1}{2} m_1 \dot{x}_1^2 \text{ et } T_2 = \tfrac{1}{2} m_2 \dot{x}_2^2 \Rightarrow T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 \qquad (4.2)$$

$$U_{e1} = \tfrac{1}{2} k_1 x_1^2 \text{ , } U_{e2} = \tfrac{1}{2} K (x_2 - x_1)^2 \text{ et } U_{e3} = \tfrac{1}{2} k_2 x_2^2 \qquad (4.3)$$

$$\Rightarrow \text{U}=\frac{1}{2} k_1 x_1^2 + \frac{1}{2} K (x_2 - x_1)^2 + \frac{1}{2} k_2 x_2^2 = \frac{1}{2}(k_1 + K) x_1^2 + \frac{1}{2}(k_2 + K) x_2^2 - \frac{1}{2} K x_1 x_2$$

$$L = T - U \Rightarrow L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \left(\frac{1}{2} k_1 x_1^2 + \frac{1}{2} K (x_2 - x_1)^2 + \frac{1}{2} k_2 x_2^2\right)$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \left(\dfrac{\partial L}{\partial x_1}\right) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \left(\dfrac{\partial L}{\partial x_2}\right) = 0 \end{cases} \copyright \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.4)$$

$$(T) \Rightarrow \begin{cases} m_1 \ddot{x}_1 + k_1 x_1 + K(x_1 - x_2) = 0 \\ m_2 \ddot{x}_2 + k_2 x_2 + K(x_2 - x_1) = 0 \end{cases} \Rightarrow \begin{cases} m_1 \ddot{x}_1 + (k_1 + K) x_1 - K x_2 = 0 \\ m_2 \ddot{x}_2 + (k_1 + K) x_2 - K x_1 = 0 \end{cases} \qquad (4.5)$$

Coupling concept:

The terms $-K_{x\,2}$ and $-K_{x\,1}$ which appear respectively in the first and second equation are called coupling terms, and the two differential equations are said to be coupled.

Solving the system of differential equations

Let us look for a particular solution of the form:

$$x_1(t) = A_1 \cos(\omega\, t + \varphi)$$

$$x_2(t) = A_2 \cos(\omega\, t + \varphi)$$

where A_1, A_2 and φ are constants and ω one of the natural pulsations of the system .

We find:

$$\begin{cases}[k_1 + K - m_1\omega^2]A_1 - K\,A_2 = 0 \\ -K\,A_1 + [k_2 + K - m_2\omega^2]A_2 = 0\end{cases} \quad (4.6)$$

Which constitutes a system of homogeneous linear equations whose unknowns are A_1 and A_2. This system admits a non-identically zero solution only if the determinant $\Delta(\omega)$ of the coefficients of A_1 and A_2 is equal to zero.

$$\Delta(\omega) = \begin{vmatrix}(k_1 + K - m_1\omega^2) & -K \\ -K & (k_2 + K - m_2\omega^2)\end{vmatrix} \quad (4.7)$$

$\Delta(\omega)$ is called the characteristic determinant.

The equation $\Delta(\omega) = 0$ is called the characteristic equation or the natural pulsation equation. It is written:

$$(k_1 + K - m_1\omega^2)(k_2 + K - m_2\omega^2) - K^2 = 0 \quad (4.8)$$

$$\Rightarrow \omega^4 - \omega^2\left(\frac{k_1+K}{m_1} + \frac{k_2+K}{m_2}\right) + \frac{k_1k_2+k_1K+k_2K}{m_1m_2} = 0 \quad (4.9)$$

This equation admits two positive real solutions ω_1 and ω_2 called the proper pulsations of the system. Each of the coordinates, x_1 and x_2, has two harmonic components of pulsations ω_1 and ω_2

$$x_1 = A_{11}\cos(\omega_1 t + \phi_1) + A_{12}\cos(\omega_2 t + \phi_2) \quad (4.10)$$
$$x_2 = A_{21}\cos(\omega_1 t + \phi_1) + A_{22}\cos(\omega_2 t + \phi_2) \quad (4.11)$$

Where $A_{11}, A_{12}, A_{21}, A_{22}, \phi_1$ et ϕ_2 are constantes

ω_1 corresponds to the smallest pulsation and ω_2 corresponds to the largest, the lowest frequency term is called the fundamental . The other term is called the harmonic.

When $A_{12} = A_{22} = 0$, x_1 and x_2 corresponding to the first particular solution are sinusoidal functions, in phase, of pulsation ω_1; the system is said to oscillate in the first mode. In this case:

$$x_1 = A_{11}\cos(\omega_1 t + \phi_1) \quad (4.12)$$
$$x_2 = A_{21}\cos(\omega_1 t + \phi_1) \quad (4.13)$$

Let us study the particularities of these two particular solutions:

The first particular solution is written:

$$\left.\begin{aligned} x_1 &= A_{11}\cos(\omega_1 t + \phi_1) \\ x_2 &= A_{21}\cos(\omega_1 t + \phi_1) \end{aligned}\right\} \Rightarrow \begin{cases} (k_1 + K - m_1\omega_1^2)A_{11} - KA_{21} = 0 \\ -KA_{11} + (k_2 + K - m_2\omega_1^2)A_{21} = 0 \end{cases} \quad (4.14)$$

$$\Rightarrow \mu_1 = \frac{A_{21}}{A_{11}} = \frac{k_1 + K - m_1\omega_1^2}{K} = \frac{K}{k_2 + K - m_2\omega_1^2}$$

The second particular solution is written:

$$\left.\begin{aligned} x_1 &= A_{12}\cos(\omega_2 t + \phi_2) \\ x_2 &= A_{22}\cos(\omega_2 t + \phi_2) \end{aligned}\right\} \Rightarrow \begin{cases} (k_1 + K - m_1\omega_2^2)A_{12} - KA_{22} = 0 \\ -KA_{12} + (k_2 + K - m_2\omega_2^2)A_{22} = 0 \end{cases} \quad (4.15)$$

$$\Rightarrow \mu_2 = \frac{A_{22}}{A_{12}} = \frac{k_1 + K - m_1\omega_2^2}{K} = \frac{K}{k_2 + K - m_2\omega_2^2} \quad (4.16)$$

The general solution (x_1 , x_2) is a linear combination of these two solutions

particular. X_1 and x_2 are then written:

$$x_1 = A_{11}\cos(\omega_1 t + \phi_1) + A_{12}\cos(\omega_2 t + \phi_2) \quad (4.17)$$
$$x_2 = \mu_1 A_{11}\cos(\omega_1 t + \phi_1) + \mu_2 A_{12}\cos(\omega_2 t + \phi_2) \quad (4.18)$$

Where $A_{11}, A_{12}, A_{21}, A_{22}, \phi_2$ et ϕ_2 are constantes .

4.2.2 Special case of two identical coupled pendulums

For this case: $m_1 = m_2 = m$, and $k_1 = k_2 = k$.

We obtaine: $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{k+2K}{m}} = \omega_1\sqrt{1+\frac{2K}{k}}$, $\mu_1 = +1$ and $\mu = -1$ (4.19)

we take $x_{10}, x_{20}, \dot{x}_{10}$ and $\dot{x}_{20}$ as 64odelled valeurs.

These initial conditions into account , we obtain the following system of equations:

$$\begin{cases} A_{11}\cos(\phi_1) + A_{12}\cos(\phi_2) = x_{10} \\ A_{11}\cos(\phi_1) - A_{12}\cos(\phi_2) = x_{20} \\ -\omega_1 A_{11}\sin(\phi_1) - \omega_2 A_{12}\sin(\phi_2) = \dot{x}_{10} \\ -\omega_1 A_{11}\sin(\phi_1) + \omega_2 A_{12}\sin(\phi_2) = \dot{x}_{20} \end{cases} \quad (4.20)$$

The solutions to this system of equations are:

$$A_{11} = \frac{x_{10}+x_{20}}{2\cos(\phi_1)} \text{ and } A_{12} = \frac{x_{10}-x_{20}}{2\cos(\phi_2)} \text{ or } A_{11} = -\frac{\dot{x}_{10}+\dot{x}_{20}}{2\omega_1\sin(\phi_1)} \text{ and}$$

$$A_{12} = -\frac{\dot{x}_{10}-\dot{x}_{20}}{2\omega_1\sin(\phi_2)}$$

1- For a particular case $x_{10} = x_{20} = x_0$ et $\dot{x}_{10} = \dot{x}_{20} = 0$ we have : $\phi_1 = \phi_2 = 0$ et $A_{12} = 0$ et $A_{11} = x_0$.

$$x_1 = x_0\cos(\omega_1 t) \quad (4.21)$$

$$x_2 = x_0\cos(\omega_1 t) \quad (4.21)$$

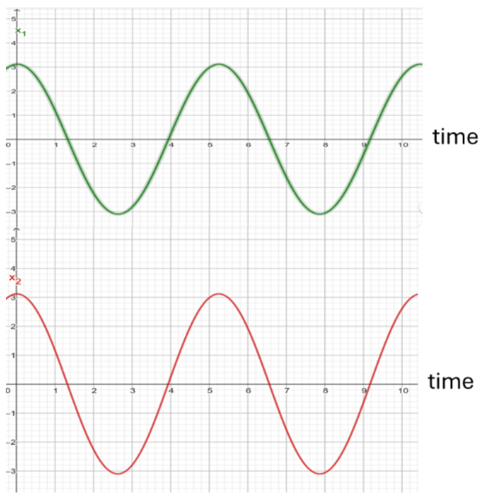


Fig.3: The system oscillates in the first mode.

For these particular initial conditions, the two masses oscillate in phase at the same pulsation ω_1. ***The system is said to oscillate in the first mode.***

4. For a particular case $x_{10} = -x_{20} = x_0$ et $\dot{x}_{10} = \dot{x}_{20}$=0:

we obtain: $\phi_1 = \phi_2$=0 et $A_{11} = 0$ et $A_{12}=x_0$, which means

$$x_1 = x_0\, cos(\,\omega_2\, t) \quad (4.23)$$

$$x_2 = -x_0\, cos(\,\omega_2\, t) \quad (4.24)$$

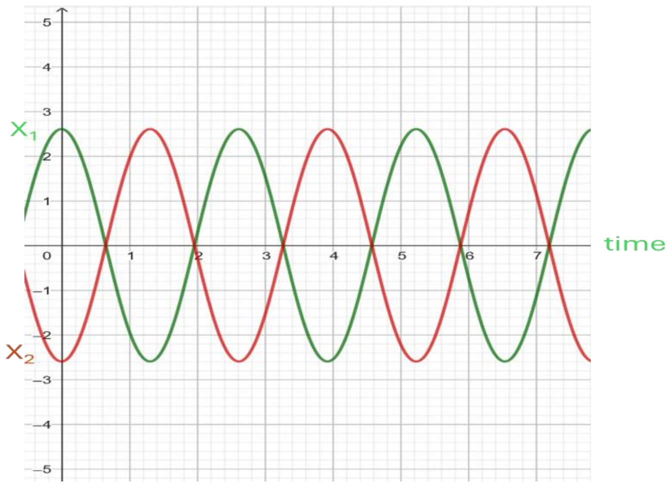


Fig.4: The system oscillates in the second mode.

For these particular initial conditions, the two masses oscillate in phase opposition at the same pulsation ω_2 . ***<u>The system is said to oscillate in the second mode.</u>***

3- Finally, let us consider the following special case:

If $x_{10} = x_0, x_{20}$ =0 et $\dot{x}_{10} = \dot{x}_{20}$=0

We obtain $\phi_1 = \phi_2$=0 et $A_{11}=A_{12} = \dfrac{x_0}{2}$

$$\begin{cases} x_1 = \frac{x_0}{2} cos(\,\omega_1\, t) + \frac{x_0}{2} cos(\,\omega_2\, t) \\ x_2 = \frac{x_0}{2} cos(\,\omega_1\, t) - \frac{x_0}{2} cos(\,\omega_2\, t) \end{cases} \Rightarrow \begin{cases} x_1 = x_0\, cos(\frac{\omega_2-\omega_1}{2} t)\, cos(\frac{\omega_2+\omega_1}{2} t) \\ x_2 = x_0\, sin(\frac{\omega_2-\omega_1}{2} t)\, sin(\frac{\omega_2+\omega_1}{2} t) \end{cases} \quad (4.25)$$

For K>>k

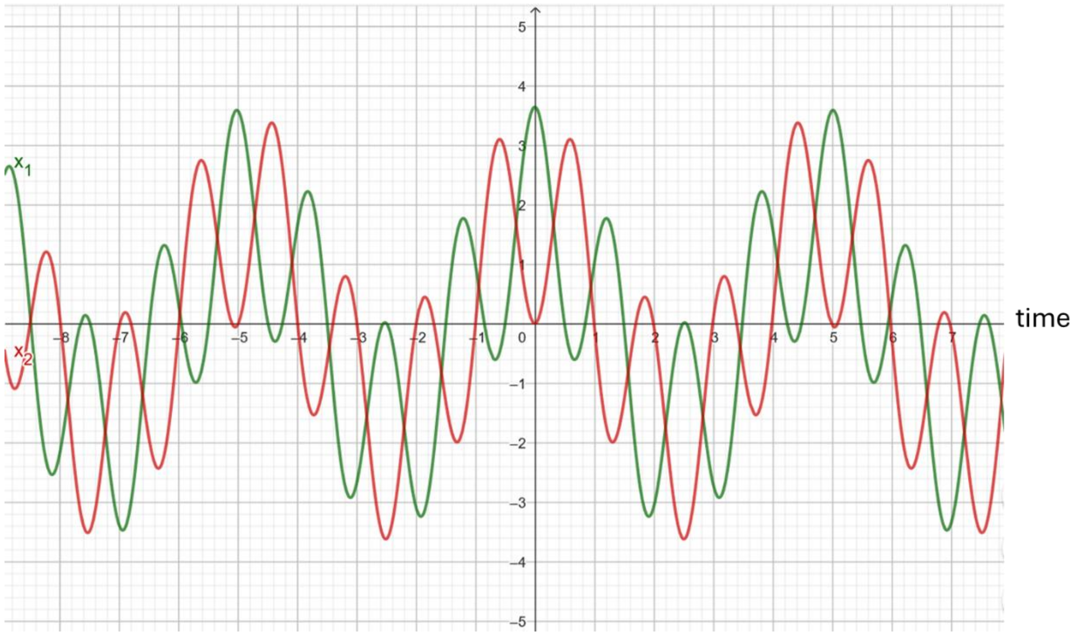


Fig.5: Oscillation of two masses when K>>k

For K<<k

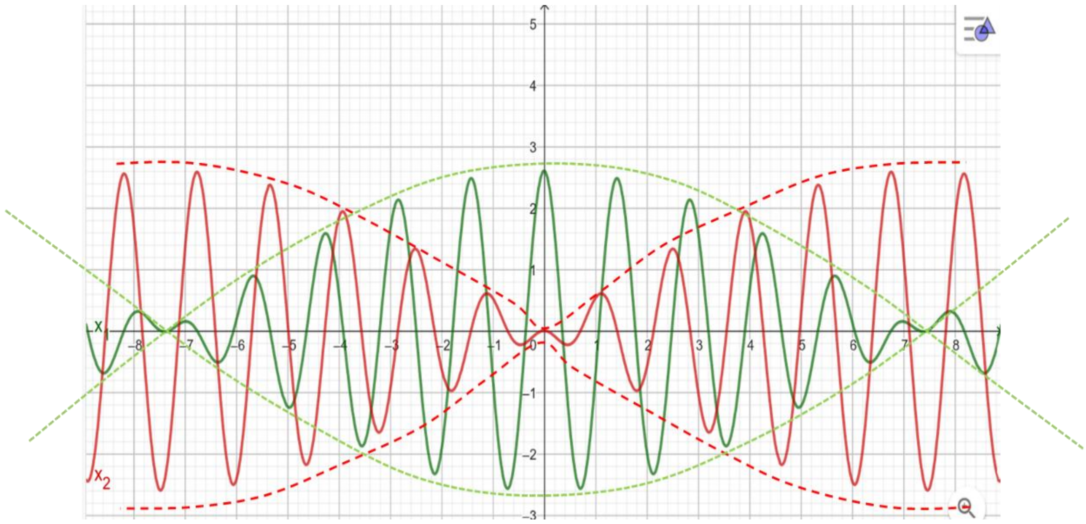


Fig.6: The beating phenomenon

The beating phenomenon demonstrates the concept of **coupled modes** and how energy can oscillate between different parts of a system. In real-world applications, this concept is significant in various areas, such as molecular vibrations in chemistry, musical instruments, and engineering structures, where coupled oscillators are common.

In chemistry, for example, molecular vibrational modes can couple and exhibit energy transfer similar to that of coupled pendulums. Understanding the beating phenomenon helps in analyzing the dynamics of such systems and how energy distributes between different vibrational modes.

Conditions for Beating:

1. **Frequency Difference**: The two angular frequencies ω_1 and ω_2 must be close to each other but not identical. This means:

   $$|\omega_1 - \omega_2| \ll \omega_1, \omega_2 \quad (4.26)$$

   In other words, the difference between $\omega_1$and ω_2 should be much smaller than the individual frequencies.

2. **Resulting Beat Frequency**: The **beat frequency** is the difference between the two frequencies:

   $$f_{beat} = |\omega_1 - \omega_2|/2\pi \quad (4.27)$$

   This represents the rate at which the amplitude of the resultant wave varies due to constructive and destructive interference.

3. **Amplitude Modulation**: The amplitude of the resulting wave oscillates between maximum (constructive interference) and minimum (destructive interference). The amplitude modulation occurs at a frequency f_{beat}.

The resulting wave has two components:

- **Fast oscillation** at the average frequency $\omega_1+\omega_2$, which determines the main oscillatory motion.
- **Slow amplitude modulation** at the difference frequency $\omega_1-\omega_2$, which causes the beating effect.

4.2.3 **Coupled pendulums**

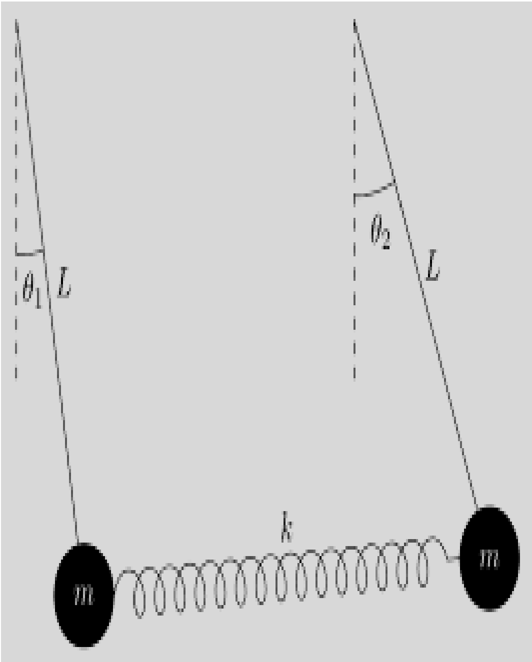


Fig.6. Coupled pendulums

Kinetic energy $T = \frac{1}{2} ml^2 \dot{\theta}_1^2 + \frac{1}{2} ml^2 \dot{\theta}_2^2$ (4.28)

Potential energy $U = \frac{1}{2}(Kl^2 + mgl)\theta_1^2 + \frac{1}{2}(Kl^2 + mgl)\theta_2^2 - Kl^2\theta_1\theta_2$ (4.29)

We note the presence of the coupling term ($- K l_2 \theta_1 \theta_2$) in the expression of the potential energy. As in the previous example, we say that the coupling is elastic. If the coupling term only exists in the expression of the kinetic energy, we say that the coupling is of the inertial type .

$$L = T - U \Rightarrow L = \frac{1}{2} ml^2 \dot{\theta}_1^2 + \frac{1}{2} ml^2 \dot{\theta}_2^2 - \frac{1}{2}(Kl^2 + mgl)\theta_1^2 - \frac{1}{2}(Kl^2 + mgl)\theta_2^2 + Kl^2\theta_1\theta_2$$

Equations of motion:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} ml^2 \ddot{\theta}_1^2 + (Kl^2 + mgl)\theta_1 - Kl^2\theta_2 = 0 \\ ml^2 \ddot{\theta}_2^2 + (Kl^2 + mgl)\theta_2 - Kl^2\theta_1 = 0 \end{cases} \quad (4.30)$$

$$\theta_1(t) = A_1 \cos(\omega t + \varphi) \quad (4.31)$$
$$\theta_2(t) = A_2 \cos(\omega t + \varphi) \quad (4.32)$$

These two expressions must satisfy the system of differential equations, hence:

$$\begin{cases}(Kl^2 + mgl - ml^2\omega^2)A_1 - Kl^2A_2 = 0 \\ -Kl^2A_1 + (Kl^2 + mgl - ml^2\omega^2)A_1 = 0\end{cases} \qquad (4.35)$$

This system of equations admits non-zero solutions only if ω is the solution to the equation at frequencies:

$$(Kl^2 + mgl - ml^2\omega^2)^2 - (Kl^2)^2 = 0 \Rightarrow \begin{cases}\omega_1 = \sqrt{\frac{g}{l}} \\ \omega_2 = \sqrt{\frac{g}{l} + \frac{2K}{m}}\end{cases} \qquad (4.36)$$

The solution to the system of differential equations is therefore:

$$\theta_1 = A_{11}\,cos(\,\omega_1\,t + \phi_1) + A_{12}\,cos(\,\omega_2\,t + \phi_2) \qquad (4.37)$$

$$\theta_2 = \mu_1 A_{11}\,cos(\,\omega_1\,t + \phi_1) + \mu_{22}A_{12}\,cos(\,\omega_2\,t + \phi_2) \qquad (4.38)$$

After the calculation we find: $\boldsymbol{\mu_{11} = 1}$, $\boldsymbol{\mu_{22} = -1}$ (4.39)

As a result,

$$\theta_1 = A_{11}\,cos(\,\omega_1\,t + \phi_1) + A_{12}\,cos(\,\omega_2\,t + \phi_2) \qquad (4.40)$$

$$\theta_2 = A_{11}\,cos(\,\omega_1\,t + \phi_1) - A_{12}\,cos(\,\omega_2\,t + \phi_2) \qquad (4.41)$$

Supplementary Exercises

Problem 1:

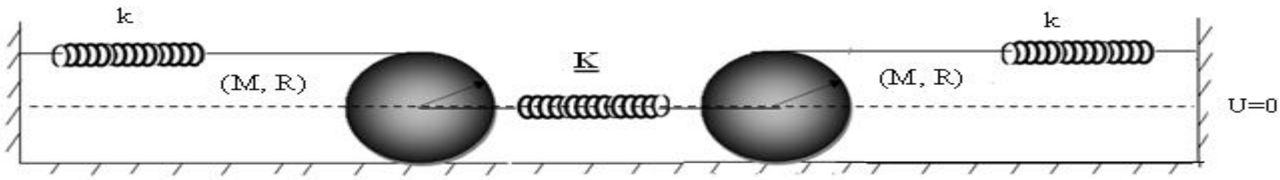
We consider the system in the figure below consisting of two identical elastic pendulums where the mass M is a solid disk of radius R connected by a spring of stiffness k. The two disks roll without sliding on a horizontal plane, a spring of stiffness **K** couples the two disks. At equilibrium the three springs are at rest.

1- How many degrees of freedom are there?
2- Give the formula of
   2-a- Kinetic energy.
   2-b- Potential energy.
   3-c- Lagrangian of the system.
3- Establish the differential equations of motion.
4- Give the expression of $\mu = \frac{A_2}{A_1}$.
5- Calculate $\omega_1 \quad et \quad \omega_2$.

6- Give the equations of motion as a function of A $_{11}$, A $_{12,}$ ω_1 *et* ω_2.

7- What is the phase shift between the two masses if the system oscillates in the second mode?

We recall that the moment of inertia of a solid disk is I $_{CM} = \frac{1}{2}MR^2$.



Solution:

$$s = 3N - n \; On\ a\ N = 2, n \begin{cases} Z_1 = 0 \\ Z_2 = 0 \\ Y_1 = 0 \\ Y_2 = 0 \end{cases} => n = 4\ donc: S = 2,$$

$$two\ degrees\ of\ freedom\ and\ two\ generalized\ coordinates\ X_1, X_2\ ou\ \theta_1, \theta_2\ .$$

Potential energy:

$$U = \frac{1}{2}k(2X_1)^2 + \frac{1}{2}k(2X_2)^2 + \frac{1}{2}K(X_1 - X_2)^2 \;\; ou\ U$$
$$= \frac{1}{2}k(2R\theta_1)^2 + \frac{1}{2}k(2R\theta_2)^2 + \frac{1}{2}K(R\theta_1 - R\theta_2)^2$$

Kinetic energy:

$$T = \frac{1}{2}m\dot{X}_1^{\ 2} + \frac{1}{2}mR^2\dot{\theta}_1^{\ 2} + \frac{1}{2}m\dot{X}_2^{\ 2} + \frac{1}{2}mR^2\dot{\theta}_2^{\ 2}$$

$$T = m\dot{X}_1^{\ 2} + m\dot{X}_2^{\ 2}\ ou\ T = mR^2\dot{\theta}_1^{\ 2} + mR^2\dot{\theta}_2^{\ 2}$$

Lagrangian of the system:

$$L = m\dot{X}_1^{\ 2} + m\dot{X}_2^{\ 2} - \frac{1}{2}(4k + K)X_1^2 - \frac{1}{2}(4k + K)X_2^2 + KX_1X_2\ ou$$

$$L = mR^2\dot{\theta}_1^{\ 2} + mR^2\dot{\theta}_2^{\ 2} - \frac{1}{2}(4k + K)R^2\theta_1^2 - \frac{1}{2}(4k + K)R^2\theta_2^2 + KR^2\theta_1\theta_2$$

The differential equations of motion:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \left(\frac{\partial L}{\partial X_1}\right) = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{X}_2}\right) - \left(\frac{\partial L}{\partial X_2}\right) = 0 \end{cases} =$$

$$> \begin{cases} 2m\ddot{X}_1 + (4k+K)X_1 - KX_2 = 0 \\ 2m\ddot{X}_2 + (4k+K)X_2 - KX_1 = 0 \end{cases} \quad or \quad \begin{cases} 2m\ddot{\theta}_1 + (4k+K)\theta_1 - K\theta_2 = 0 \dots\dots (1) \\ 2m\ddot{\theta}_2 + (4k+K)\theta_2 - K\theta_1 = 0 \dots\dots (2) \end{cases}$$

$\theta_1(t) = A_1 \cos(\omega t + \varphi)$ et $\theta_2(t) = A_2 \cos(\omega t + \varphi)$

$$(1) => -2m\omega^2 A_1 \, cos(\omega t + \varphi) + (4k+K)A_1 \, cos(\omega t + \varphi) - KA_2 \, cos(\omega t + \varphi) = 0$$

$$=> [(4k+K) - 2m\omega^2]A_1 \, cos(\omega t + \varphi) - KX_2 = 0 \;=>\; X_2 = \frac{[(4k+K) - 2m\omega^2]}{K} A_1 \, cos(\omega t + \varphi)$$

$$\mu = \frac{[(4k+K) - 2m\omega^2]}{K}$$

$$=> \begin{cases} [(4k+K) - 2m\omega^2]A_1 - KA_2 = 0 \\ -KA_1 + [(4k+K) - 2m\omega^2]A_2 = 0 \end{cases}$$

$$=> det \begin{vmatrix} [(4k+K) - 2m\omega^2] & -K \\ -K & [(4k+K) - 2m\omega^2] \end{vmatrix} = 0 => \begin{cases} \omega_1^2 = \frac{2k}{m} \\ \omega_2^2 = \frac{2k+K}{m} \end{cases}$$

$$=> \begin{cases} \mu_1 = \frac{[(4k+K) - 2m\omega_1^2]}{K} \\ \mu_2 = \frac{[(4k+K) - 2m\omega_2^2]}{K} \end{cases} => \begin{cases} \mu_1 = \frac{A_{21}}{A_{11}} = 1 \\ \mu_2 = \frac{A_{22}}{A_{21}} = -1 \end{cases}$$

$$\Rightarrow \begin{cases} X_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\,cos(\omega_2 t + \varphi_2) \\ X_2(t) = A_{11}\,cos(\omega_1 t + \varphi_1) - A_{12}\,cos(\omega_2 t + \varphi_2) \end{cases}$$

$$or \begin{cases} \theta_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\,cos(\omega_2 t + \varphi_2) \\ \theta_2(t) = A_{11}\,cos(\omega_1 t + \varphi_1) - A_{12}\,cos(\omega_2 t + \varphi_2) \end{cases}$$

The phase shift if the system oscillated in second mode W_2

$$=> \begin{cases} X_1(t) = A_{12}\,cos(\omega_2 t + \varphi_2) \\ X_2(t) = -A_{12}\,cos(\omega_2 t + \varphi_2) \end{cases} => \begin{cases} X_1(t) = A_{12}\,cos(\omega_2 t + \varphi_2) \\ X_2(t) = A_{12}\,cos(\omega_2 t + \varphi_2 + \pi) \end{cases}$$

The phase shift between X_1 and $X_2 = \pi$

Chapter 5: General Information on Propagation Phenomena

1. Introduction

In this chapter, we will explore the fundamental principles governing the propagation of waves, a phenomenon that is central to many physical processes. Wave propagation describes how disturbances travel through various media, transferring energy without permanent displacement of the medium itself. Understanding these principles is essential for comprehending a wide array of chemical and physical phenomena, including sound waves in fluids, light waves in optics, and molecular vibrations.

The chapter will introduce key concepts such as wave speed, wavelength, frequency, and amplitude, as well as the different types of waves-mechanical and electromagnetic. Special attention will be given to the mathematical representation of waves using wave equations, which serve as powerful tools for analyzing wave behavior in different contexts.

In the context of chemistry, wave propagation plays a significant role in understanding how energy moves through molecules, how sound waves are used in analytical techniques, and how light interacts with matter. Whether studying acoustic waves in liquids or electromagnetic waves in spectroscopy, the ability to understand and model wave behavior is crucial to interpreting experimental data and advancing chemical research.

Through this chapter, students will gain a foundational understanding of the mechanisms behind wave propagation, preparing them for more specific discussions of acoustic waves and optical phenomena in subsequent sections.

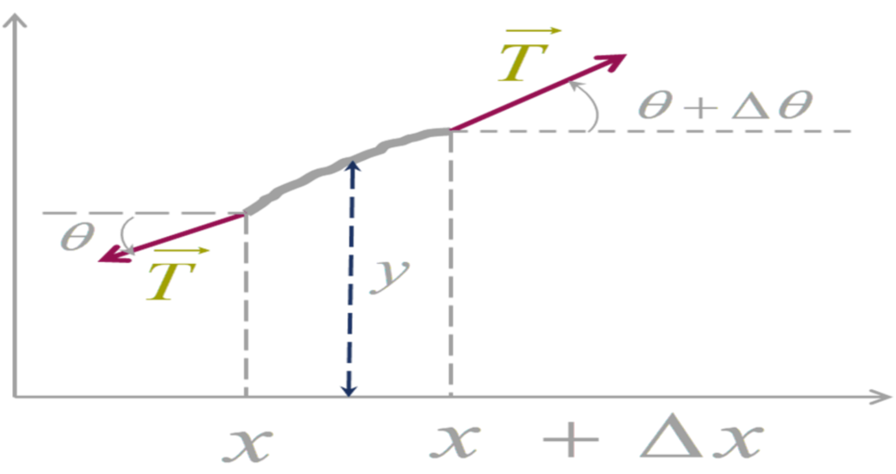


Fig.1: propagation wave along a string

2. Propagation Equation (Wave Equation)

The fundamental equation governing wave propagation in one dimension is the wave equation. For a wave traveling along the xx-axis, this equation can be written as:

$$\frac{\partial^2 \mathrm{u(x,t)}}{\partial \mathrm{x}^2} = \frac{1}{v^2}\frac{\partial^2 \mathrm{u(x,t)}}{\partial t^2} \quad (5.1)$$

where: u(x,t) is the wave function representing the displacement of the wave at position x and time t, v is the speed of wave propagation.

This is a second-order partial differential equation, describing how the wave's shape evolves over time and space.

3. Simple case propagation wave along a string:

Consider a taut rope, straight along the x coordinate, and of infinite length . We will study the propagation of a weak shock along the rope. Let us suppose that this shock occurs along the Oy axis (see Fig1).

$\sum \vec{F} = \overrightarrow{T_1} + \overrightarrow{T_2}$ (5.2)

We have no movement along the Ox axis,

So: $F_y = -T\,sin(\theta) + T\,sin(\theta + \Delta\theta) = T\Delta\theta$ (5.3)

$dm.\ddot{y} = T\Delta\theta \quad Or \quad dm = \mu.\Delta x$ (5.4)

$where\ \mu$ is the mass density of the string

In the other side $tan(\theta) = \frac{\partial y}{\partial x} \Rightarrow \frac{1}{cos^2(\theta)}\frac{\partial \theta}{\partial x} = \frac{\partial^2 y}{\partial x^2} \Rightarrow \Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x$ (5.4)

$\mu\Delta x\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\Delta x \Rightarrow \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$ où $v = \sqrt{\frac{T}{\mu}}$ (5.5)

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \quad (5.6)$$

Equation (5.6) is the wave equation of the string or wave propagation

4. Solution of the Propagation Equation

A general solution to the one-dimensional wave equation can be written as:

$$u(x,t)=f(x-vt)+g(x+vt) \quad (5.7)$$

where:

$f(x-vt)$ represents a wave traveling to the right (in the positive x-direction) with velocity **v**,

$g(x+vt)$ represents a wave traveling to the left (in the negative xx-direction) with velocity **v**.

These are known as progressive waves. Depending on the initial conditions, one of these terms may dominate or both may coexist, representing waves moving in both directions.

5. **Sinusoidal Progressive Wave**

A specific and important type of solution is the sinusoidal progressive wave, which describes a wave with a harmonic (sinusoidal) form. A sinusoidal wave traveling to the right can be expressed as:

$$u(x,t)=A\sin(kx-\omega t+\phi) \quad (5.8)$$

where:

A is the amplitude of the wave (maximum displacement),

k is the wave number (related to the wavelength),

ω is the angular frequency (related to the frequency),

ϕ is the phase constant, determining the initial phase of the wave.

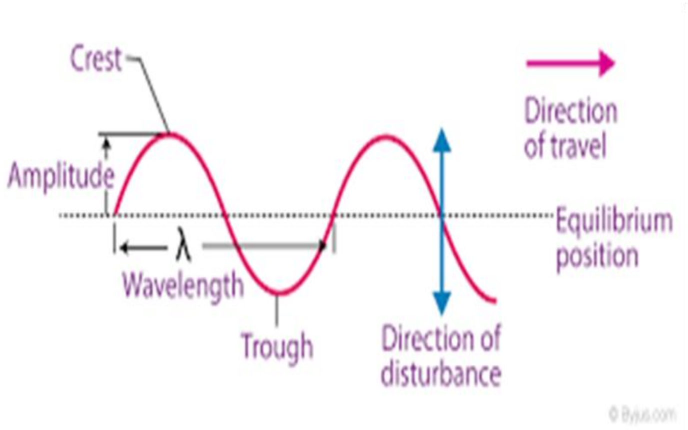


Fig.2. Sinusoidal progressive wave

5.1.Wavelength and Wave Number

The wavelength λ is the distance between successive points of the wave that are in phase (e.g., between two peaks or troughs). It is related to the wave number k by the following relationship:

$$k = \frac{2\pi}{\lambda} \quad (5.9)$$

The wave number k represents the number of wavelengths per unit distance and is measured in radians per meter.

5.2.Frequency and Angular Frequency

The frequency f of the wave, which is the number of oscillations per second, is related to the angular frequency ω by:

$$\omega=2\pi f \quad (5.10)$$

The relationship between wave speed v, frequency f, and wavelength λ is given by:

$$v=f\lambda \quad (5.11)$$

This equation expresses that the wave speed is the product of the frequency and the wavelength.

5.3.Example: Sinusoidal Wave Propagation

Consider a specific example of a wave traveling along a string with a wave speed v=300 m/s, a wavelength λ=0.5 m, and a frequency f=600 Hz. The wave equation can be written as:

$$u(x,t)=A\sin(2\pi(0.5-600t))$$

For this wave:

A is the amplitude,

$k=2\pi/\lambda=2\pi/0.5=12.57$ rad/ m,

$\omega=2\pi f=2\pi\times 600=3769.91$ rad/s

The wave travels to the right at 300 m/s and repeats every 0.5 malong the x-axis.

6. Linear Chain Model: Study and Mathematical Development

The Linear Chain Model is commonly used in physics and chemistry to describe the vibrational properties of atoms or molecules arranged in a periodic structure, such as atoms in a solid or a polymer chain. The model assumes that particles are connected by harmonic springs, which mimic interatomic or intermolecular forces. This simplified model provides insights into collective vibrational behavior (phonons) and is essential for understanding the dynamics of solids, molecular vibrations, and crystal lattices.

Below is a detailed study of the Linear Chain Model, focusing on its mathematical formulation and key concepts.

7.1.Physical Model

Consider a linear chain of **N** identical particles (atoms or molecules), each with mass mm, connected by identical springs with spring constant k. The particles can oscillate along the x-axis, and the restoring force between adjacent particles obeys Hooke’s law.

For simplicity, we assume that:

- The particles are constrained to move in one dimension (along the chain),

- The displacements of the particles from their equilibrium positions are small, allowing us to treat the system as harmonic.

Let $u_n(t)$ represent the displacement of the n^{-th} particle from its equilibrium position at time t.

7.2.Equation of Motion

For each particle in the chain, Newton's second law applies. The force on the n_n^{-th} particle due to its two nearest neighbors is given by Hooke's law, assuming linear restoring forces. The equation of motion for the n_n^{-th} particle is:

$$m\frac{d^2u_n(t)}{dt^2} = k[u_{n+1}(t) - u_n(t)] - k[u_n(t) - u_{n-1}(t)] \quad (5.12)$$

This equation expresses the net force on the n^{-th} particle as the difference between the forces from its right neighbor (n+1) and left neighbor (n−1).

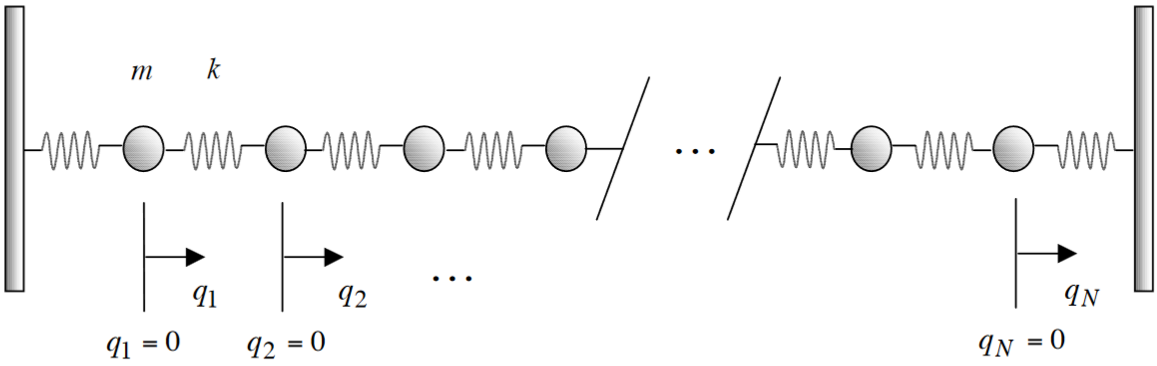


Fig.3. Linear chain model

Equation (5.12) is a second-order difference equation in space and a second-order differential equation in time, representing the collective motion of the chain.

7.3.Solution by Traveling Waves

To solve the equation of motion, we assume that the displacements $u_n(t)$ take the form of a plane wave solution, which is common for periodic systems. The wave-like solution is:

$$u_n(t) = Ae^{i(qn-\omega t)} \quad (5.13)$$

where:

A is the amplitude of oscillation,

q is the wave vector, related to the wavelength of the wave,

n indexes the particle along the chain, and,

ω is the angular frequency of oscillation.

7.4.Dispersion Relation

Substituting the assumed solution $u_n(t) = Ae^{i(qn-\omega t)}$ into the equation of motion, we get:

$$m\frac{d^2}{dt^2}\left(Ae^{i(qn-\omega t)}\right) = kA\left(e^{i(q(n+1)-\omega t)} + e^{i(q(n-1)-\omega t)} - 2e^{i(qn-\omega t)}\right) \quad (5.14)$$

- The left-hand side becomes:

$$-m\omega^2 Ae^{i(qn-\omega t)} \quad (5.15)$$

The right-hand side involves displacements of the neighboring particles:

$$kA(e^{iq} + e^{-iq} - 2)e^{i(qn-\omega t)} \quad (5.16)$$

Since $e^{iq} + e^{-iq} = 2cos(q)$, the equation becomes:

$$-m\omega^2 Ae^{i(qn-\omega t)} = kA(2cos(q) - 2)e^{i(qn-\omega t)} \quad (5.17)$$

Canceling out the common factors (including the exponential terms), we get the following relation between ω and q:

$$-m\omega^2 = kA(2cos(q) - 2)$$

$$\omega^2 = \frac{2kA}{m}\left(1 - cos(q)\right) \quad (5.18)$$

Finally, we can express the angular frequency ω as:

$$\omega(q) = 2\sqrt{\frac{k}{m}}\left|\sin\left(\frac{q}{2}\right)\right| \qquad (5.19)$$

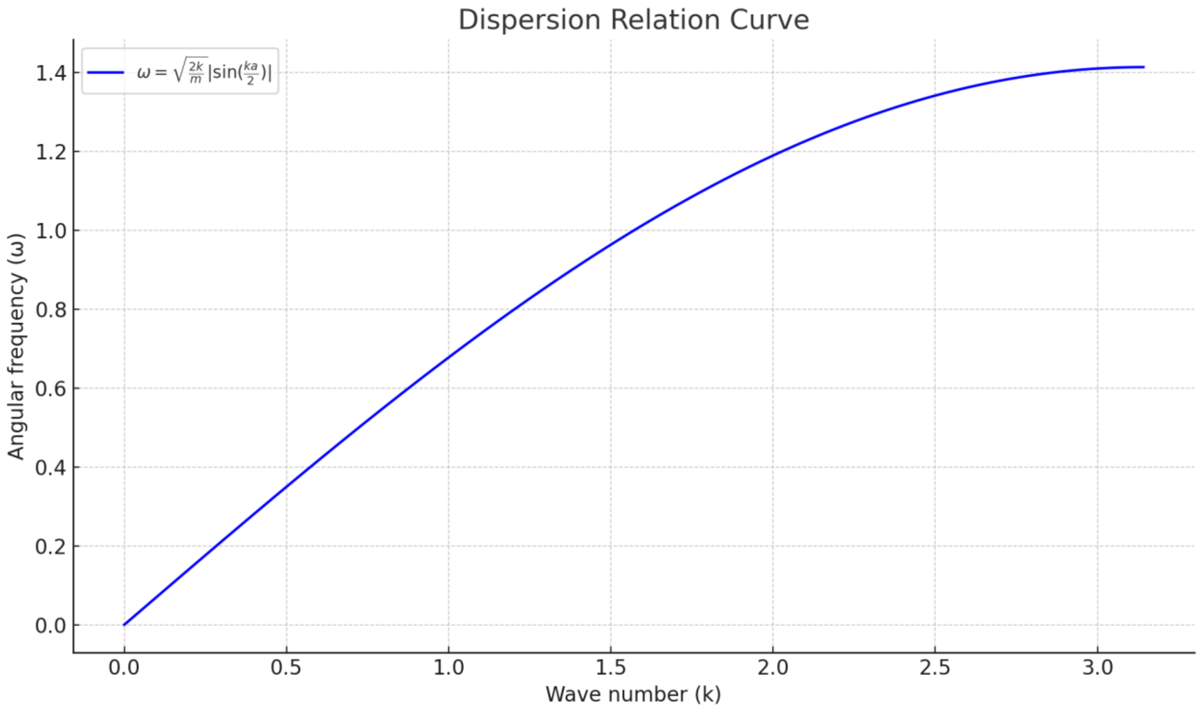


Fig.4. Curve of dispersion relation

This is the key result for the linear chain model, showing how the angular frequency ω of the wave depends on the wave vector q. The dispersion relation tells us how the frequency of the normal modes varies with the wave vector q.

7.5.**Normal Modes and Phonons**

Each value of q corresponds to a normal mode of the system, representing a collective oscillation of all the particles in the chain. These normal modes are the vibrational states of the system, and in the quantum mechanical description, they correspond to quantized vibrations called phonons. Phonons are crucial in understanding thermal properties, heat conduction, and vibrational spectroscopy in solid-state systems.

Test your comrhension:

A source of vibration at one end of a string under tension has a displacement given by the equation

S(0,t)= 0.1sin(6t) , where **S** is in meters and t is in seconds. The tension of the rope is 4N **and** its mass per unit length is **μ= 0.01Kg.m** $^{-1}$.

- What is the speed of propagation of the wave in the rope?
- What is the frequency of the wave?
- What is its wavelength?
- What is the equation for the displacement of a point located at 1 meter from the source?
- What is the particle speed of a point located at 3 meters from the source?

Solution:

A source of vibration at one end of a string under tension has a displacement given by the equation

S (0, t) = 0.1 sin (6t) , where **S** is in meters and t is in seconds. The tension in the rope is **4N** and its mass per unit length is **μ = 0.01Kg.m** $^{-1}$.

- What is the speed of propagation of the wave in the rope?

Answer: $V=\sqrt{\frac{T}{\mu}} = 20 m/S.$

- What is the frequency of the wave?

Answer: $f = 2\pi\omega = 0.95$ **Hz.**

- What is its wavelength?

Answer: λ=20/(0.95)=20.93m.

- What is the equation for the displacement of a point located 1 meter from the source?

$$S(1, t) = 0.1 \sin[6(t - \frac{1}{20})]$$

- What is the particle speed of a point located 3 meters from the source?

$$\dot{S}(3, t) = 0.6 \cos [6 (t - \frac{3}{20})]$$

Chapter 6: Acoustic Waves in Fluids

1. Introduction

Acoustic waves, commonly known as sound waves, are mechanical disturbances that propagate through a medium by inducing vibrations of the particles within it. In fluids (gases and liquids), these waves travel by compressing and expanding regions of the medium, generating alternating high and low-pressure zones that move in the direction of wave propagation. The study of acoustic waves in fluids is fundamental to understanding a variety of physical phenomena, from sound transmission in air and water to pressure waves in gases during industrial and chemical processes.

Acoustic waves in fluids can be described as longitudinal waves, where the motion of fluid particles occurs in the same direction as the wave. Unlike transverse waves, where displacement occurs perpendicular to the direction of propagation, acoustic waves in fluids are governed by compressions and rarefactions along the wave's path. This chapter focuses on the mathematical and physical principles governing acoustic wave propagation in fluids, including the derivation of the wave equation, wave speed, and key characteristics such as frequency, wavelength, and pressure variations.

We will begin by exploring the fundamental equations that describe fluid dynamics, such as the continuity equation and the Euler equation, and how they combine to form the acoustic wave equation. The chapter also covers important concepts such as the speed of sound in various fluids, the nature of pressure variations in acoustic waves, and the effects of factors like temperature and fluid density on wave propagation. Additionally, we will examine the practical applications of acoustic waves, ranging from sound wave transmission in air to sonar in underwater exploration.

By the end of this chapter, students will have a comprehensive understanding of how acoustic waves propagate in fluids, the mathematical tools used to analyze these waves, and their practical relevance in fields such as acoustics, engineering, and environmental sciences.

This chapter delves into the behavior and mathematical description of acoustic waves propagating through fluids (gases and liquids). The principles of wave propagation are essential in understanding sound dynamics, which have practical applications in fields like chemical sensing and reaction monitoring. The chapter is structured around three key topics: the propagation equation for acoustic waves, characteristics of progressive sinusoidal waves, and reflection-transmission phenomena at normal incidence.

2. Equation of Propagation of Acoustic Waves in Fluids and Speed of Sound

The propagation of sound waves in fluids is governed by the wave equation, which is derived from the fundamental principles of fluid dynamics and thermodynamics. Sound is a mechanical wave that results from oscillations in pressure and density, transmitted through the medium due to particle interactions.

3. Wave Equation in Fluids

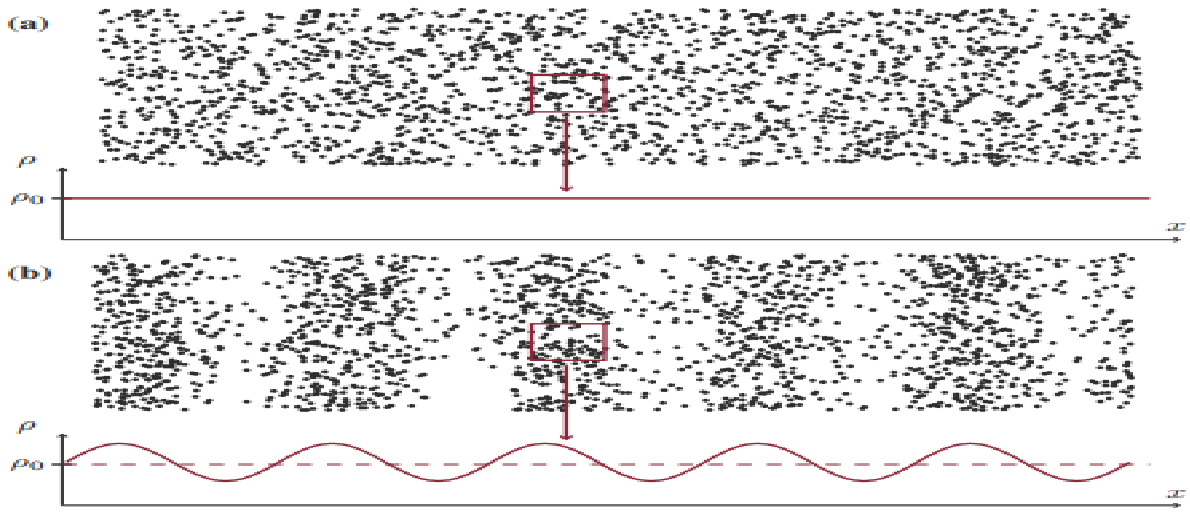


Fig.1:Fluid at rest and subjected to a compression wave. At equilibrium (a), the molecules of the fluid are randomly distributed throughout the available volume, and the density r(x) = r0 is homogeneous. When a wave passes (b), high density areas and low density areas are created

Acoustic waves in fluids can be described by the **wave equation**, which is derived from three basic equations:

3.1.Continuity equation (mass conservation),

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0 \quad (6.1)$$

For small perturbations, we approximate this as:

$$\frac{\partial \grave{\rho}}{\partial t} + \rho_0 \nabla.(v) = 0 \quad (6.2)$$

Where:

$\grave{\rho}$ is the small change in the density from the equilibrium density ρ_0, and v is the fluid velocity due to the wave.

3.2.Euler's equation (momentum conservation),

Euler's equation describes momentum conservation in the fluid. For small perturbations, it is written as:

$$p_0 \frac{\partial v}{\partial t} = -\nabla \grave{p} \quad (6.3)$$

Where $\grave{p}$ is the small perturbation in pressure from the equilibrium pressure p_0.

3.3.Equation of state (thermodynamic relation between pressure and density)

The equation of state relates the pressure perturbation $\grave{p}$ to the density perturbation $\grave{\rho}$. For an ideal gas, this is often written as:

$$\grave{p} = c_s^2 \grave{\rho} \quad (6.4)$$

Where c_S is the speed of sound in the fluid.

3.4.Derivation of the Wave Equation

To derive the **wave equation** for pressure or density perturbations, we combine the continuity and Euler equations. Taking the time derivative of the continuity equation and substituting the velocity v from Euler's equation:

3.4.a. Differentiate the continuity equation with respect to time:

$$\frac{\partial^2 \grave{\rho}}{\partial t^2} = -\rho_0 \nabla . \frac{\partial v}{\partial t} \quad (6.5)$$

Use Euler s equation: $\frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \nabla \grave{p}$ to substitude for $\frac{\partial v}{\partial t}$:

$$\frac{\partial^2 \grave{\rho}}{\partial t^2} = \nabla^2 \grave{p} \quad (6.6)$$

Using the equation of state $\grave{p} = c_s^2\grave{\rho}$, substitute for $\grave{p}$ to get

$$\frac{\partial^2\grave{\rho}}{\partial t^2} = c_s^2\nabla^2\grave{\rho} \quad (6.7)$$

This is the acoustic wave equation for density perturbations:

$$\frac{\partial^2\grave{\rho}}{\partial t^2} - c_s^2\nabla^2\grave{\rho} = 0 \quad (6.8)$$

A similar equation can be written for pressure perturbations:

$$\frac{\partial^2\grave{p}}{\partial t^2} - c_s^2\nabla^2\grave{p} = 0 \quad (6.9)$$

4. Speed of sound c_s

The speed pf sound c_s in fluid depends on both the compressibility and the density of the fluid. In general, the speed of sound is given by:

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} \quad (6.10)$$

Where:

γ is the adiabatic index (the ratio of specific heats C_p/C_v)

P_0 is the equilibrium pressure,

ρ_0 is the equilibrium density

4.a. Speed of sound in gases

For an ideal gas, the speed of sound is commonly expressed as:

$$c_s = \sqrt{\gamma\frac{RT}{M}} \quad (6.11)$$

Where:

R is the universal gas constant, T is the temperature of the gas, and M is the molar mass of the gas.

For dry air at room temperature (approximately 20°C), the speed of sound is about 343m/s.

4.b. Speed of sound in liquids

In liquids, the speed of sound depends on the ***bulk modulus B*** and the density ρ_0 :

$$c_s = \sqrt{\frac{B}{\rho_0}} \qquad (6.12)$$

The bulk modulus B is measure of the fluid's resistance to compression. For water at room temperature, the speed of sound is about 1480 m/s.

Let's work through a couple of examples to calculate the **speed of sound** in different media

Examples:

Example 1: Speed of sound in air (at 20°C)

At room temperature, air behaves like an ideal gas. We can use the formula for the speed of sound in ideal gas:

$$c_s = \sqrt{\gamma \frac{RT}{M}}$$

Where

$\gamma = 1.4$ (adiabatic index for air)

R=8.314 J/mol (universal gas constant)

T = 293K (temperature, 20°C)

M=0.029kg/mol (molar mass of air).

Now, plug in the values:

$$c_s = \sqrt{1.4x\frac{8.314x293}{0.029}} \approx 342.89m/s$$

Thus, the speed of sound in air at 20°C is approximately **343 m/s**.

Example 2: Speed of sound in water (at 25°C)

For liquids like water, the speed of sound depends on the bulk modulus B and the density ρ_0 :

$$c_s = \sqrt{\frac{B}{\rho_0}}$$

At 25 °C:

The bulk modulus of water $B \approx 2.2x10^9$Pa, and the density of water $\rho_0 \approx 1000kg/m^3$.

Now, plug in the values:

$$c_s = \sqrt{\frac{2.2x10^9}{10^3}} \approx 1483m/s$$

Example 3: Speed of sound in steel (at 25°C)

For solids like steel, the speed of sound is determined by the Young's modulus E and the density ρ of the material:

$$c_s = \sqrt{\frac{E}{\rho}} \qquad (6.13)$$

For steel in 25°C: Young's modulus$E \approx 2.1x10^{11}Pa,$ Density $\rho \approx 7850kg/m^3$

$$c_s = \sqrt{\frac{2.1x10^{11}}{7850}} \approx 5174m/s$$

Thus, the speed of sound in steel is approximately 5174m/s.

These examples highlight how the speed of sound varies significantly across different media due to defferences in their density and elastic properties.

Chapter 7: Principles and Laws of Geometric Optics

1. Introduction:

Geometric optics is a branch of optics that describes the propagation of light as rays, enabling a simplified analysis of light behavior as it interacts with various optical elements, such as lenses, mirrors, and apertures. This approach is particularly useful for understanding how light travels in a straight line, reflects off surfaces, and refracts through different media. Unlike wave optics, which accounts for the wave nature of light and its associated phenomena such as interference and diffraction, geometric optics operates under the assumption that light travels in linear paths, known as rays.

This chapter delves into the fundamental principles and laws governing geometric optics, beginning with the concept of light propagation and the characteristics of light rays. We will explore **Snell's Law**, which describes how light bends when transitioning between media of different refractive indices, and the **law of reflection**, which states that the angle of incidence equals the angle of reflection. These principles form the foundation for understanding optical phenomena in various applications, from simple magnifying glasses to complex optical systems used in modern technologies.

Furthermore, this chapter and the next chapter will cover critical optical devices such as mirrors and lenses, elucidating how they manipulate light to form images. We will analyze the behavior of converging and diverging lenses, including the construction of ray diagrams to illustrate image formation and the concepts of focal length and magnification.

By the end of this chapter, students will have a thorough understanding of the principles and laws of geometric optics, equipping them with the knowledge to analyze and design optical

2. Reflection

The reflection of light occurs when a light ray strikes a surface and bounces back into the original medium. The behavior of reflected light can be described using two main principles: the law of reflection and the geometry of ray diagrams.

3. Law of Reflection

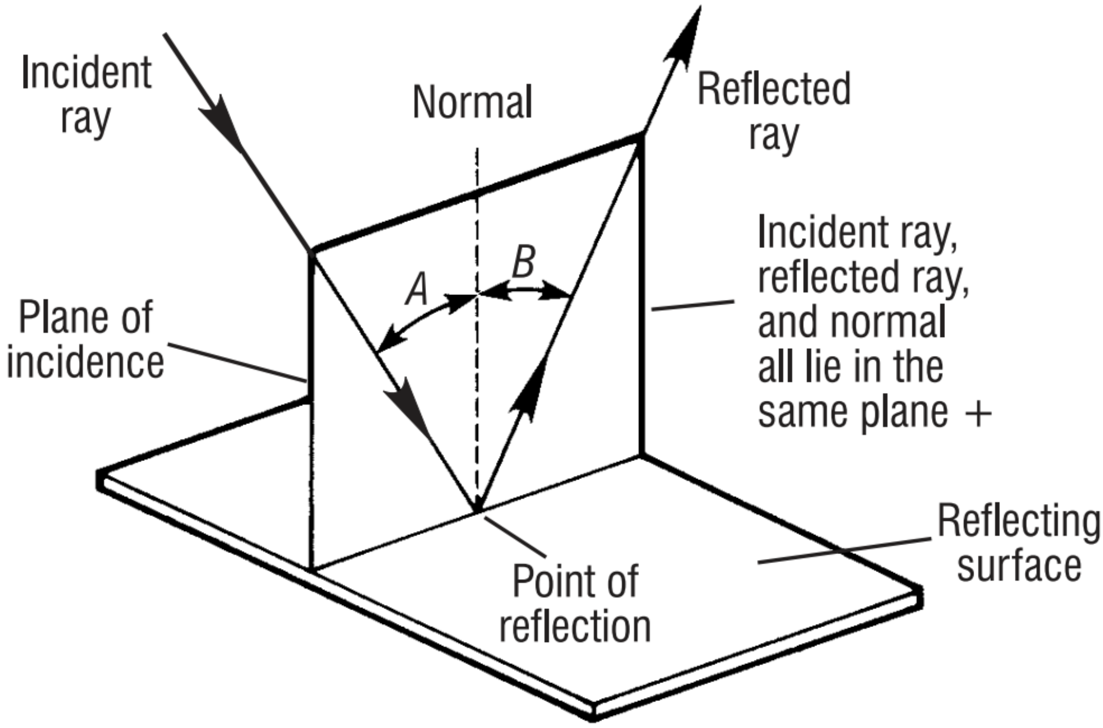


Fig1. *Law of reflection: Angle B equals angle A.*

The law of reflection states that:

The angle of incidence (θi) is equal to the angle of reflection (θr).

Mathematically, this can be expressed as:

$$\theta_i=\theta_r \quad (7.1)$$

Where:

θ_i is the angle between the incident ray and the normal (perpendicular) to the surface.

θ_r is the angle between the reflected ray and the normal (see Fig1).

4. Refraction

Refraction is the bending of light as it passes from one medium into another with a different refractive index. The law of refraction, commonly known as Snell's Law, governs this phenomenon.

(a) Snell's Law

Snell's Law relates the angles of incidence and refraction to the refractive indices of the two media. It is mathematically expressed as:

$$n_1\sin(\theta_i)=n_2\sin(\theta_r) \quad (7.2)$$

Where:

n_1 is the refractive index of the first medium.

N_2 is the refractive index of the second medium.

θ_i is the angle of incidence.

θ_r is the angle of refraction.

(b) Refractive Index

The refractive index **n** of a medium is defined as:

$$\mathbf{n=c/v}$$

Where:

c is the speed of light in a vacuum (approximately 3×10^8 m/s).

v is the speed of light in the medium.

For example, the refractive index of water is approximately nwater≈1.33nmeaning light travels slower in water than in a vacuum.

(c) Ray Diagram for Refraction

To illustrate refraction using a ray diagram:

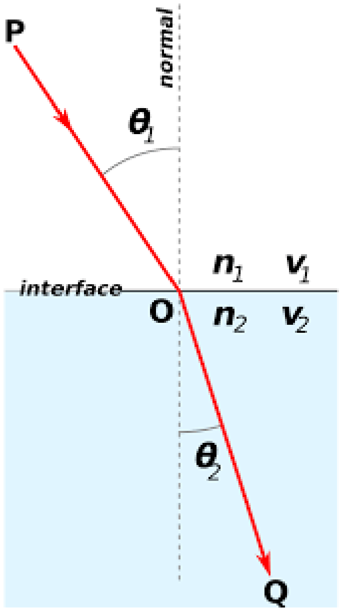


Fig2. Ray Diagram for Refraction

Draw the incident ray approaching the interface between the two media at angle θi.

Draw the normal line at the point of incidence. Measure the angle θr in the second medium, ensuring that it follows Snell's Law.

5. Refraction through a Prism

Definition of Prisms

Prisms are optical elements that refract light, causing it to disperse into its constituent colors. The behavior of light passing through a prism can be analyzed using Snell's Law and geometric principles.

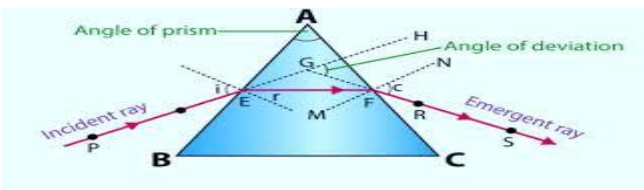


Fig3. Refraction through a Prism

When light enters and exits a prism, it undergoes two refractions. For a prism with an apex angle A, the following relationships can be derived:

Angle of Incidence at the First Surface: Let θ_1 be the angle of incidence at the first face of the prism, and n be the refractive index of the prism material.

Using Snell's Law at the first interface:

$$n_1 \sin(\theta_1) = n \sin(\theta_2)$$

Where:

n_1 is the refractive index of the incident medium (usually air, where $n1 \approx 1$).

θ_2 is the angle of refraction into the prism.

Angle of Refraction at the Second Surface: When the light exits the prism, let θ_3 be the angle of incidence at the second face, and θ_4 be the angle of refraction in the air.

Using Snell's Law at the second interface:

$$n \sin(\theta_3) = n_1 \sin(\theta_4)$$

Relating Angles in the Prism:

The relationship between the angles can be expressed as:

$$\theta_1 + \theta_2 + \theta_4 = A$$

Using these equations, one can calculate the angles of incidence and refraction as light passes through the prism.

(b) **Dispersion of Light**

Prisms can also be used to demonstrate dispersion, which occurs when different wavelengths of light are refracted by different amounts due to their varying speeds in the prism material. The relationship between the angles of refraction and wavelength can be expressed as:

$$\Delta\theta=\theta_r(\text{violet})-\theta_r(\text{red})$$

This angle difference leads to the separation of white light into its constituent colors when passed through a prism, creating a spectrum.

Example 1: Refraction (Air to Water)

Problem: A light ray passes from air into water at an angle of incidence of 45°. The refractive index of air is approximately $n_1=1.0$and for water, $n_2=1.33$. What is the angle of refraction?

Solution: Using Snell's Law:

$$n_1\sin(\theta i)=n_2\sin(\theta r)$$

Substituting the known values:

$$1.0\cdot\sin(45°)=1.33\cdot\sin(\theta r)$$

Calculating sin(45°):

$$\sin(45°)=22\approx0.707$$

So:

$$\sin(\theta r)=1.330.707\approx0.532$$

Now calculating θrθr:

$$\theta r\approx\arcsin(0.532)\approx32.1°$$

Conclusion: The angle of refraction is approximately 32.1∘.

Example 2: Refraction Through a Prism

Problem: A light ray enters a prism with a refractive index of n=1.5 at an angle of incidence of 60∘. The apex angle of the prism is A=45°. What is the angle of refraction as the light exits the prism?

Solution:

Refraction at the First Face: Using Snell's Law:

$$n_1 \sin(\theta_1) = n2 \sin(\theta_2)$$

Where $n_1=1.0$, $\theta_1=60\circ$, and $n_2=1.5$.

$$1.0 \cdot \sin(60\circ) = 1.5 \cdot \sin(\theta 2)$$

Calculating sin(60∘):

$$\sin(60\circ) = 32 \approx 0.866$$

So:

$$0.866 = 1.5 \cdot \sin(\theta_2)$$

$$\sin(\theta_2) = 0.8661.5 \approx 0.577$$

Now calculating θ2θ2:

$$\theta_2 \approx \arcsin(0.577) \approx 35.0°$$

Refraction at the Second Face: The angle of incidence at the second face can be calculated as:

$$\theta_3 = \theta_2 + A = 35.0° + 45° = 80°$$

Using Snell's Law again for the exit face:

$$n_2 \sin(\theta_3) = n_1 \sin(\theta_4)$$

Where $n_1 = 1.0$ (air) and $n_2 = 1.5$:

$$1.5 \cdot \sin(80°) = 1.0 \cdot \sin(\theta_4)$$

Calculating sin(80°):

$$\sin(80°) \approx 0.985$$

So:

$$1.5 \cdot 0.985 = \sin(\theta_4)$$

$$\sin(\theta_4) \approx 1.477$$

Since $\sin(\theta_4) > 1$, this means total internal reflection occurs, and the light will not exit the prism.

Conclusion: The light does not exit the prism due to total internal reflection.

Chapter 8: Construction of Images

1. Introduction:

The construction of images is a fundamental concept in optics that pertains to how optical systems, such as lenses and mirrors, form visual representations of objects. Understanding the principles governing image formation is essential for students of optics, as it lays the groundwork for numerous applications in fields ranging from photography and microscopy to optical engineering and vision science.

In this chapter, we will explore the various types of optical elements—particularly concave and convex lenses, as well as concave and convex mirrors—and their roles in image formation. We will begin by establishing the basic concepts of object distance, image distance, and focal length, which are critical to the understanding of how images are constructed.

The chapter will delve into the use of ray diagrams, a graphical method for predicting the location and characteristics of images formed by optical devices. We will analyze the behavior of light rays as they interact with different surfaces and media, applying the principles of reflection and refraction to illustrate the formation of real and virtual images.

Additionally, we will explore the characteristics of images, including size, orientation, and type (real or virtual), while also considering the magnification produced by optical systems. Understanding these characteristics is crucial for practical applications, such as in the design of lenses for cameras, microscopes, and corrective eyewear.

Through mathematical models and practical examples, this chapter will provide a comprehensive overview of how images are constructed in optical systems, equipping students with the knowledge to analyze and design various optical instruments. By the end of this chapter, students will have a solid grasp of the principles of image formation, enabling them to apply these concepts to real-world scenarios in science and technology.

Theoretical Background for the Chapter: Construction of Images

This chapter explores the fundamental principles of image formation through various optical elements. We will cover stigma, plane and spherical diopters, plane and spherical mirrors, and thin lenses. Each section will provide detailed explanations alongside visual aids to facilitate understanding.

2. Stigma

Stigma refers to the quality of an optical system in producing clear and sharp images. The main objective in optics is to minimize aberrations to achieve optimal stigma.

3. Types of Aberrations

Spherical Aberration: Occurs when light rays that strike a spherical lens or mirror at different distances from the optical axis converge at different points. This causes the image to become blurred, as not all rays focus at the same point.

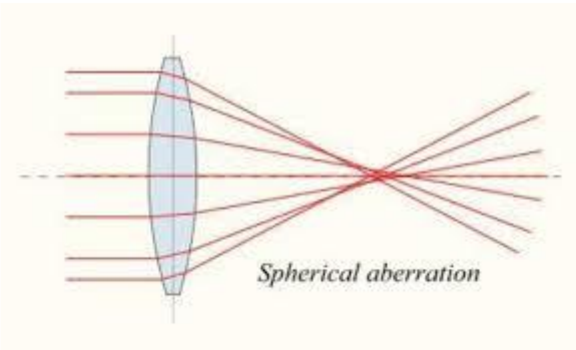


Figure 1: Spherical Aberration

This diagram illustrates how parallel rays of light converge at various points due to their different distances from the optical axis, leading to a blurred image.

4. Spherical Aberration Diagram

Chromatic Aberration: This type occurs because different wavelengths of light are refracted by varying amounts when passing through a lens. This results in color fringing, where different colors focus at different points.

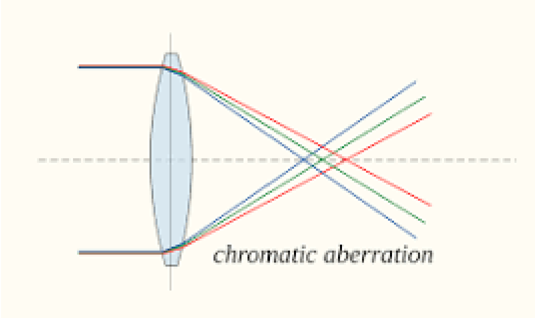


Figure 2: Chromatic Aberration

This diagram demonstrates how a lens disperses different wavelengths of light, showing the varying focal points for each color.

5. Chromatic Aberration Diagram

To achieve optimal stigma, various techniques, such as using aspheric lenses or multi-element systems, can be employed to minimize these aberrations.

6. Plane and Spherical Diopters

Diopters are optical elements that bend light rays. The power D of a lens or mirror is defined as:

$$D=1/f \quad (8.1)$$

Where:

D is the power in diopters (D).

f is the focal length in meters (m).

(a) **Plane Diopters**

Plane Mirrors: These reflect light without changing the convergence of rays. The image formed by a plane mirror is virtual, upright, and the same size as the object. The distance from the object to the mirror is equal to the distance from the image to the mirror.

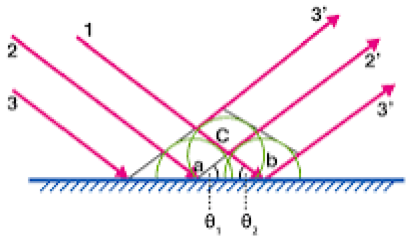


Figure 3: Reflection in a Plane Mirror

This ray diagram illustrates how incoming light rays reflect off a plane mirror, demonstrating the law of reflection, where the angle of incidence equals the angle of reflection.

(b) Spherical Diopters

Spherical Mirrors: These can be concave or convex. The focal length f for spherical mirrors is related to their radius of curvature R:

$$f=R/2 \qquad (8.2)$$

Concave Mirrors: These mirrors converge light rays. When the object is placed outside the focal length, real and inverted images are formed.

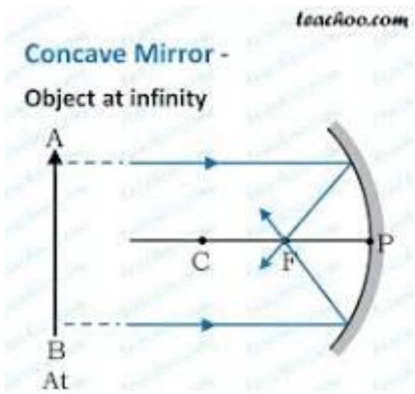


Figure 4: Concave Mirror Diagram

This diagram shows how a concave mirror converges light rays to a focal point, resulting in the formation of a real image.

Convex Mirrors: These mirrors diverge light rays, forming virtual images that are smaller and upright. The image appears behind the mirror.

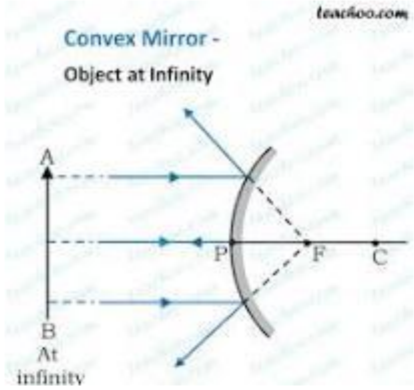


Figure 5: Convex Mirror Diagram

This ray diagram illustrates how a convex mirror diverges incoming light rays, leading to the formation of a virtual image.

(C) Plane and Spherical Mirrors

Mirrors are classified into two main categories: plane mirrors and spherical mirrors.

(a) Plane Mirrors

Reflection Principle: The law of reflection states that the angle of incidence equals the angle of reflection. As a result, the image formed by a plane mirror is virtual, upright, and of the same size as the object. The distance from the object to the mirror is equal to the distance from the image to the mirror.

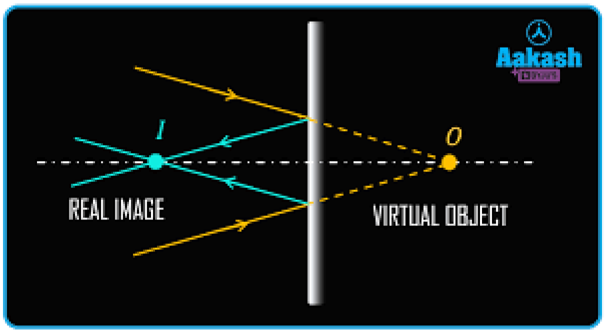


Figure 6: relationship between the object, the plane mirror, and the virtual image formed

This diagram highlights the relationship between the object, the plane mirror, and the virtual image formed.

(D) Spherical Mirrors

Concave Mirrors: These mirrors can form both real and virtual images depending on the object's distance relative to the focal point.

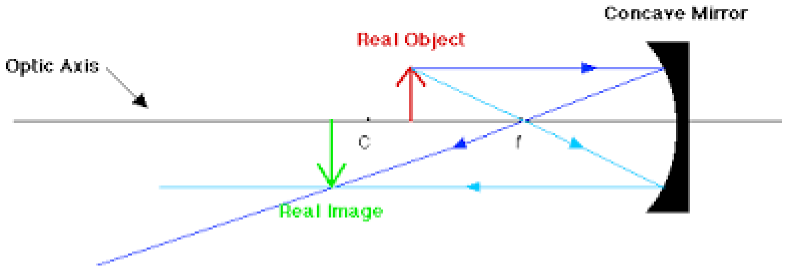


Figure 7: Real Image Formation by Concave Mirror

This diagram shows the process of image formation by a concave mirror, indicating how a real image is produced when the object is beyond the focal point.

Convex Mirrors: Form virtual images that appear smaller and upright.

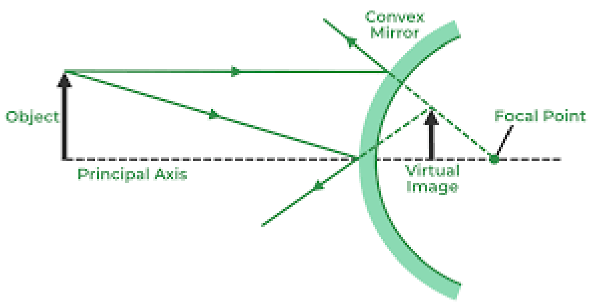


Figure 8: Virtual Image Formation by Convex Mirror

This diagram demonstrates how a convex mirror diverges light rays, leading to a virtual image behind the mirror.

(E)Thin Lenses

Thin lenses are critical components in optical systems, utilized to either converge or diverge light rays. They can be categorized as convex lenses or concave lenses.

(a) Convex Lenses

Convergence: Convex lenses converge parallel rays of light to a single focal point. Depending on the object's position relative to the focal length, a real inverted image or a virtual upright image can be formed.

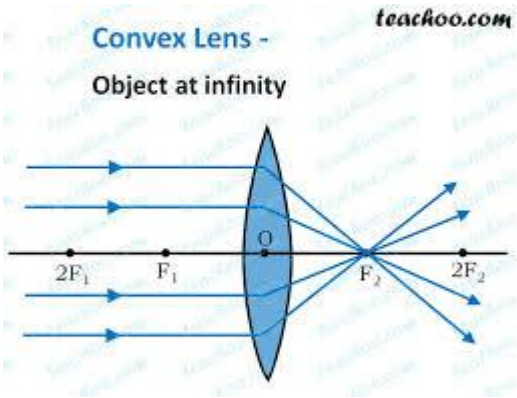


Figure 9: Convex Lens Diagram

This ray diagram illustrates how a convex lens focuses parallel rays to a focal point, leading to the formation of a real image.

The thin lens formula is given by:

$$1/f=1/d_o+1/d_i \quad (8.4)$$

Where:

(d_o) is the object distance (positive when measured from the lens).

(d_i) is the image distance (positive for real images and negative for virtual images).

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