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Course Notes

Equations of Mathematical Physics

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Chapter 1

Generalities about partial differential equations

In this chapter we first introduce certain basic concepts and definitions of partial differential equations (PDEs).

1.1 What is a differential equation?

An *ordinary differential equation* (ODE) is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function:

$$F(t, u(t), u'(t), u^{(2)}(t), u^{(3)}(t), \dots, u^{(m)}(t)) = 0.$$

This is an example of an ODE of order m where m is a highest order of the derivative in the equation. Solving an equation like this on an interval $t \in [0, T]$ would mean finding a function $t \mapsto u(t) \in \mathbb{R}$ with the property that u and its derivatives satisfy this equation for all values $t \in [0, T]$.

The problem can be enlarged by replacing the real-valued u by a vector-valued one $u(x, y) = (u_1(t), u_2(t), \dots, u_N(t))$. In this case we usually talk about system of ODEs. Even in this situation, the challenge is to find functions depending upon exactly one variable which, together with their derivatives, satisfy the equation.

1.1.1 What is a partial derivative?

When you have function that depends upon several variables, you can differentiate with respect to either variable while holding the other variable constant. This spawns the idea of partial derivatives. As an example, consider a function depending upon two real variables taking values in the reals:

$$u : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

As $n = 2$ we sometimes visualize a function like this by considering its graph viewed as a surface in \mathbb{R}^3 given by the collection of points $\{(x, y, z) \in \mathbb{R}^3 : z = u(x, y)\}$. We can calculate the derivative with respect to x while holding y fixed. This leads to u_x , also expressed as $\partial_x u$, $\frac{\partial u}{\partial x}$, and $\frac{\partial}{\partial x} u$. Similarly, we can hold x fixed and differentiate with respect to y .

1.2 What is PDE?

A partial differential equation (PDE) is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yx}(x, y), u_{yy}(x, y)) = 0.$$

This is an example of a PDE of order 2. Solving an equation like this would mean finding a function $(x, y) \mapsto u(x, y)$ with the property that u and its derivatives satisfy this equation for all admissible arguments.

Similarly to ODE case this problem can be enlarged by replacing the real-valued u by a vector-valued one $u(x, y) = (u_1(x, y), u_2(x, y), \dots, u_N(x, y))$. In this case we usually talk about system of PDEs.

1.2.1 Where PDEs are coming from?

PDEs are often referred as *Equations of Mathematical Physics* (or *Mathematical Physics* but it is incorrect as Mathematical Physics is now a separate field of mathematics) because many of PDEs are coming from different domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc). However PDEs appear in other fields of science as well (like quantum chemistry, chemical kinetics); some PDEs are coming from economics

and financial mathematics, or computer science. Many PDEs are originated in other fields of mathematics.

1.2.2 Some examples of PDEs

Some examples of the PDEs are given by:

Simplest first order equation: $u_x = 0$.

Transport equation: $u_t + cu_x = 0$.

Laplaces equation (in 2D): $\Delta u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) = 0$ or similarly in the higher dimensions.

Heat equation: $u_t = k\Delta u$. (The expression Δ is called the Laplacian (Laplace operator) and is defined as $\partial_x^2 + \partial_y^2 + \partial_z^2$ on \mathbb{R}^3).

Wave equation: $u_{tt} - c^2\Delta u = 0$, sometimes $\square = c^{-2}\partial_t^2 - \Delta$ is called (*d'Alembertian or d'Alembert operator*).

It appears in elasticity, acoustics, electromagnetism and so on.

One-dimensional wave equation $u_{tt}(x, t) - c^2u_{xx}(x, t) = 0$ often is called string equation and describes a vibrating string.

Oscillating rod or plate (elasticity) Equation of vibrating rod (with one spatial variable) $u_{tt} + Ku_{xxx} = 0$ or vibrating plate (with two spatial variables) $u_{tt} + K\Delta^2 u = 0$.

Navier-Stokes equation (hydrodynamics for incompressible liquid)

$$\begin{cases} \rho v_t + (v \cdot \nabla)\rho v - \nu \Delta v = -\nabla p \\ \nabla \cdot v = 0, \end{cases}$$

where ρ is a (constant) density, v is a velocity and p is the pressure; when viscosity $\nu = 0$ we get Euler equation

$$\begin{cases} \rho v_t + (v \cdot \nabla)\rho v = -\nabla p \\ \nabla \cdot v = 0. \end{cases}$$

Einstein equation for general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}$ is the *Einstein tensor*, $g_{\mu\nu}$ is the *metric tensor* (unknown functions), $R_{\mu\nu}$ is the *Ricci curvature tensor*, and R is the scalar curvature, $T_{\mu\nu}$ is the stressenergy tensor, Λ is the cosmological constant and κ is the Einstein gravitational constant. Components of Ricci

curvature tensor are expressed through the components of the metric tensor, their first and second derivatives.

1.3 Basic concepts and definitions

Definition 1. A partial differential equation (PDE) is an equation that involves an unknown function of two or more independent variables and certain partial derivatives of the unknown function. More precisely, let u denote a function of the n independent variables x_1, \dots, x_n , $n \geq 2$. Then a relation of the form

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \dots) = 0$$

where F is a function of its arguments, is a partial differential equation in u .

The following equations are some examples of partial differential equations in two independent variables x and y

$$\begin{aligned} (a) \quad & xu_x(x, y) + yu_y(x, y) - 2u(x, y) = 0. \\ (b) \quad & yu_x(x, y) - xu_y(x, y) = x. \\ (c) \quad & u_{xx}(x, y) - u_y(x, y) - u(x, y) = 0. \\ (d) \quad & u(x, y)u_x(x, y) + yu_y(x, y) - u(x, y) = xy^2. \\ (e) \quad & u_{xx}(x, y) + xu_y^2(x, y) + yu(x, y) = y. \end{aligned} \tag{1.1}$$

Definition 2. The highest-order derivative appearing in a partial differential equation is called *the order* of the PDE.

Definition 3. The dimension of a PDE refers to the number of independent variables present.

Definition 4. A partial differential equation is usually considered in a certain domain of the independent variables. If there exists a function u in the domain under consideration, such that u and its derivatives identically satisfy the differential equation, then u is called **a solution** of the PDE.

Example 1. Verify that the function $u(x, y) = x^2 + y^2$ is a solution of equation (a) in (1.1).

Solution : the given function can be differentiated and we find

$$u_x = 2x, \quad u_y = 2y$$

and so

$$xu_x + yu_y - 2u = 2x^2 + 2y^2 - 2(x^2 + y^2) = 0.$$

for all x and y .

Example 2. Verify that the function $u(x, y) = e^{-2y} \cos x$ is a solution of equation in (c) in (1.1).

Solution : here we have

$$u_x = -e^{-2y} \sin x, \quad u_{xx} = -e^{-2y} \cos x, \quad u_y = -2e^{-2y} \cos x.$$

Thus,

$$u_{xx} - u_y - u = 0$$

for all x and y .

1.4 General solutions and auxiliary conditions

A solution of a partial differential equation that includes every other solution of the equation is called the general solution. We recall that in the theory of ordinary differential equations, the general solution of an n th order differential equation involves n independent arbitrary constants. These constants are determined when the solution is required to satisfy certain conditions. For instance, the first-order differential equation

$$\frac{du}{dx} + u = f(x). \quad (1.2)$$

has the general solution

$$u(x) = \int_0^x e^{-(x-t)} f(t) dt + Ce^{-x}. \quad (1.3)$$

involving an arbitrary constant C . If u is required to satisfy the initial condition $u(0) = 1$, then we find $C = 1$.

In the case of a partial differential equation, the general solution is found to involve arbitrary functions rather than arbitrary constants. The number of these arbitrary functions is generally equal to the order of the differential equation. Moreover, if the differential equation involves m independent variables, the arbitrary functions will depend on $(m - 1)$ variables. Thus, if the function u in equation (1.2) also depends on the variable y , then the constant C could be chosen as an arbitrary function of y . In other words, if u is a function of the independent variables x, y , then the first-order partial differential equation

$$\frac{\partial u}{\partial x} + u = f(x). \quad (1.4)$$

has for its general solution

$$u(x, y) = \int_0^x e^{-(x-t)} f(t) dt + g(y)e^{-x}. \quad (1.5)$$

where g is an arbitrary function of y . It is easy to see that for any choice of the function g , u satisfies equation (1.4).

Example 3. Find the general solution of the second-order equation

$$u_{xy} = \sin x + y$$

Solution : the general solution can be obtained by partial integration. That is, we integrate both sides of the equation with respect to y , treating x as a constant, to obtain

$$u_x(x, y) = y \sin x + \frac{y^2}{2} + h(x)$$

h being an arbitrary function of x . Next, integrating with respect to x , treating y as a constant, we obtain

$$u(x, y) = -y \cos x + \frac{xy^2}{2} + \int h(x) dx + g(y).$$

if we write

$$f(x) = \int h(x) dx.$$

then the general solution is given by

$$u(x, y) = -y \cos x + \frac{xy^2}{2} + f(x) + g(y).$$

involving two arbitrary functions f and g .

Remark 1. For a linear ordinary differential equation, the general solution depends mainly on arbitrary constants. Un like ODEs, in linear partial differential equations, the general solution depends on arbitrary functions.

1.4.1 Differentiation of composite functions : the Chain Rule

Suppose that u is a function of the variables x and y in a domain D ,

$$u = U(x, y), \quad (x, y) \text{ in } D \quad (1.6)$$

and suppose that x and y are themselves functions of the variable t ,

$$x = f(t), \quad y = g(t) \quad (1.7)$$

such that for each t in $[a, b]$, $(f(t), g(t))$ is in D . Then, in effect, u is a function of the single variable t :

$$u = U(x, y) = U[f(t), g(t)]. \quad (1.8)$$

The derivative $\frac{du}{dt}$, if it exists, can then be calculated by simply differentiating the function (1.8) with respect to t . However, it is frequently desirable to obtain the derivative of u with respect to t without an actual substitution for the variables x and y . One important and useful tool in this connection is called the "chain rule."

Theorem 1. (The Chain Rule). *If the function u in (1.6) and its first-order partial derivatives are continuous in D , and if the functions f and g in (1.7) are differentiable in $a < t < b$, then u is a differentiable function of t whose derivative is given by the formula*

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} = \frac{\partial U}{\partial x} \frac{df}{dt} + \frac{\partial U}{\partial y} \frac{dg}{dt}. \quad (1.9)$$

Example 4. *Show that $u(x, y) = f(x - 2y) + g(x + y)$, where f and g are twice differential functions, is the general solution of the differential equation*

$$2u_{xx} - u_{xy} - u_{yy} = 0.$$

Solution : we first verify that the given function satisfies the differential equation. We have

$$u_x = f' + g', \quad u_y = -2f' + g', \quad u_{xx} = f'' + g''.$$

$$u_{xy} = -2f'' + g'', \quad u_{yy} = 4f'' + g''.$$

and so

$$2u_{xx} - u_{xy} - u_{yy} = 0.$$

To show that u is the general solution, let us introduce the new variables

$$s = x - 2y, \quad t = x + y.$$

and write $w(s, t) = u(x, y)$. Then by **Chain Rule**, we have

$$u_x = w_s + w_t, \quad u_y = -2w_s + w_t.$$

$$u_{xx} = w_{ss} + 2w_{st} + w_{tt}, \quad u_{xy} = -2w_{ss} - w_{st} + w_{tt}.$$

$$u_{yy} = 4w_{ss} - 4w_{st} + w_{tt}.$$

Substitution of these in the equation yields

$$9w_{st} = 0$$

whose general solution is $w(s, t) = f(s) + g(t)$. Thus, the general solution of the equation

$$2u_{xx} - u_{xy} - u_{yy} = 0$$

is

$$u(x, t) = f(x - 2y) + g(x + y).$$

where f and g are arbitrary functions.

1.5 Linear operators and the principle of superposition

We say that L is an operator on functions if it transforms each function u of one given class into a function denoted by Lu of another class. For example, the "wave operator".

$$L = \left(\frac{\partial}{\partial t} \right)^2 - c^2 \left(\frac{\partial}{\partial x} \right)^2. \quad (1.10)$$

transforms each function u that has second-order partial derivatives with respect to x and t into the new function

$$Lu = u_{tt} - c^2 u_{xx}$$

The particular operator (1.10) is called a partial differential operator because it involves basically the operation of partial differentiation.

Definition 5. An operator L is said to be **linear**, if for any constants c_1 and c_2 and any functions u_1 and u_2 for which both Lu_1 and Lu_2 are defined, it is true that

$$L(c_1 u_1 + c_2 u_2) = c_1 Lu_1 + c_2 Lu_2. \quad (1.11)$$

Remark 2. It follows readily from the rules of partial differentiation that the wave operator (1.10) is a linear partial differential operator. Property (1.11) can be extended immediately to any finite

number of functions by induction. That is, if u_1, \dots, u_n are n -functions and c_1, \dots, c_n are n -constants, then

$$L\left(\sum_{i=1}^n c_i u_i\right) = \sum_{i=1}^n c_i L u_i. \quad (1.12)$$

We call the function $\sum_{i=1}^n c_i u_i$ a linear combination of u_1, \dots, u_n .

Let L and M denote two linear operators. We define their sum $L + M$ as the operator defined by the equation

$$(L + M)u = Lu + Mu \quad (1.13)$$

for all functions u for which both Lu and Mu are defined.

In similar manner we define the product LM of the linear operators L and M by the equation

$$(LM)u = L(Mu). \quad (1.14)$$

for all functions u for which both Mu and $L(Mu)$ are defined. From (1.11) it follows that

$$\begin{aligned} (LM)(c_1 u_1 + c_2 u_2) &= L(c_1 M u_1 + c_2 M u_2) \\ &= c_1 (LM)u_1 + c_2 (LM)u_2. \end{aligned} \quad (1.15)$$

which establishes the fact that the product of linear operators is also linear.

Definition 6. Let L denote a **linear partial differential operator**. An equation of the form

$$Lu = f \quad (1.16)$$

where f is a given function, is called a **linear partial differential equation**. If $f = 0$, the equation (1.16) is said to be **homogeneous**; otherwise, it is called **non homogeneous**.

Definition 7. Let u_1, \dots, u_n be n -functions that satisfy the homogeneous equation

$$Lu = 0. \quad (1.17)$$

Then, by (1.11), it follows that any linear combination of these functions also satisfies equation (1.17). This important fact is known as **the principle of superposition**.

Remark 3. For a linear homogeneous ordinary differential equation, it is well-known that if

$u_1, u_2, u_3, \dots, u_n$ are solutions of the equation, then a linear combination of u_1, u_2, u_3, \dots given by

$$u = c_1u_1 + c_2u_2 + c_3u_3 + \dots + c_nu_n,$$

is also a solution. The concept of combining two or more of these solutions is called **the superposition principle**.

It is interesting to note that the superposition principle works effectively for linear homogeneous PDEs in a given domain. The concept will be explained in Chapter 4 when using the method of separation of variables.

Example 5. Find the dimension and order of the following PDEs. Which are linear, and which are homogeneous?

$$(1) u_t = Du_{xx} + f(x);$$

$$(2) u_{tt} - c^2u_{xx} = 0;$$

$$(3) u_{xx} + u_{yy} = 0;$$

$$(4) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0;$$

$$(5) \frac{\partial^2 u}{\partial x^2} + e^y \sin z \frac{\partial^2 u}{\partial x \partial z} = u;$$

$$(6) \frac{\partial^2 u}{\partial x \partial y} = \sin u;$$

$$(7) u_t + uu_{xx} + u_{xxx} = 1.$$

Solution

1. Equation (1) is a two-dimensional second-order linear nonhomogeneous (for $f \neq 0$) PDE. It is sometimes called the one-dimensional heat equation since the space variable x is one-dimensional. It can be written as $Lu = f$, with the differential operator

$$L = \frac{\partial}{\partial t} - D \frac{\partial^2}{\partial x^2}.$$

2. Equation (2) is a two-dimensional second-order linear homogeneous PDE. It is also sometimes called the one-dimensional wave equation, since, again, the space variable x is one-dimensional. It can be written as $Lu = 0$, with

$$L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}.$$

3. Equation (3) is a two-dimensional second-order linear homogeneous PDE, with

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

4. Equation (4) is a two-dimensional first-order linear homogeneous PDE, with

$$L = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

5. Equation (5) is a three-dimensional second-order linear homogeneous PDE, with

$$L = \frac{\partial^2}{\partial x^2} + e^y \sin z \frac{\partial^2}{\partial x \partial z} - 1.$$

6. Equation (6) is a two-dimensional second-order nonlinear nonhomogeneous PDE.

7. Equation (7) is a two-dimensional third-order nonlinear nonhomogeneous PDE.

1.6 Boundary conditions

As stated above, the general solution of a PDE is of little use. A particular solution is frequently required that will satisfy prescribed conditions. Given a PDE that controls the mathematical behavior of physical phenomenon in a bounded domain D , the dependent variable u is usually prescribed at the boundary of the domain D . The boundary data is called boundary conditions. The boundary conditions are given in three types defined as follows:

1. Dirichlet Boundary Conditions: In this case, the function u is usually prescribed on the boundary of the bounded domain. For a rod of length L , where $0 < x < L$, the boundary conditions are defined by $u(0) = \alpha$, $u(L) = \beta$, where α and β are constants. For a rectangular plate, $0 < x < L_1$, $0 < y < L_2$, the boundary conditions $u(0, y)$, $u(L_1, y)$, $u(x, 0)$, and $u(x, L_2)$ are usually prescribed. The boundary conditions are called homogeneous if the dependent variable u at any point on the boundary is zero, otherwise the boundary conditions are called inhomogeneous.

2. Neumann Boundary Conditions: In this case, the normal derivative $\frac{du}{dn}$ of u along the outward normal to the boundary is prescribed. For a rod of length L , Neumann boundary conditions are of the form $u_x(0, t) = \alpha$, $u_x(L, t) = \beta$.

3. Mixed Boundary Conditions: In this case, a linear combination of the dependent variable u and the normal form $\frac{du}{dn}$ is prescribed on the boundary. It is important to note that it is not always necessary for the domain to be bounded, however one or more parts of the boundary

may be at infinity. This type of problems will be discussed in the coming chapters.

1.7 Initial conditions

It was indicated before that the PDEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation phenomena and phenomena of quantum mechanics. Most of the PDEs, such as the diffusion equation and the wave equation, depend on the time t . Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. It will be discussed later that for the heat case, the initial value $u(t = 0)$, that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions $u(t = 0)$ and $u_t(t = 0)$ should also be prescribed.

1.8 Well-posed PDEs

A partial differential equation is said to be well-posed if a unique solution that satisfies the equation and the prescribed conditions exists, and provided that the unique solution obtained is stable. The solution to a PDE is said to be stable if a small change in the conditions or the coefficients of the PDE results in a small change in the solution.

1.9 Exercises

Exercise 1.1. Determine the order of each of the following equations and find which are linear and homogeneous.

$$1) \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} = y; \quad 2) \left(\frac{\partial u}{\partial x}\right)^2 + u \left(\frac{\partial u}{\partial y}\right) = 1; \quad 3) \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0;$$

$$4) \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \sin x; \quad 5) \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \sin u = e^y.$$

Exercise 1.2. Show that the function u given by $u(x, t) = \frac{1}{2} + \frac{1}{2}e^{-4kt} \cos(2x)$ is the solution of the following problem:

$$\begin{cases} u_t(x, t) - ku_{xx}(x, t) = 0 & 0 < x < \pi; t > 0, \\ u(x, 0) = \cos^2 x & 0 \leq x \leq \pi, \\ u_x(0, t) = u_x(\pi, t) = 0 & t \geq 0. \end{cases}$$

Exercise 1.3. Show that the function u given by $u(x, t) = t \sin \pi x$ is the solution of the following

problem:

$$(P) \begin{cases} u_{tt}(x, t) - u_{xx}(x, t) = \pi^2 t \sin \pi x & 0 < x < 1; t > 0 \\ u(x, 0) = 0, \quad u_t(x, 0) = \sin \pi x & 0 \leq x \leq 1 \\ u(0, t) = 0, \quad u(1, t) = 0 & t \geq 0. \end{cases}$$

Exercise 1.4. In each of the following equations, find the general solution of the equation.

- (a) $u_x(x; y) = x^2 + y^2;$
- (b) $u_y(x, y, z) = xz + yz;$
- (c) $u_{xxz}(x, y, z) = x + y - z.$

Exercise 1.5. Let (E) the partial differential equation given by

$$(x^2 + 1) \frac{\partial u}{\partial x} + 2xy \frac{\partial u}{\partial y} = 0. \tag{E}$$

Find the general solution of the equation (E) using the following change of variables

$$\xi = x \text{ et } \eta = \frac{y}{x^2 + 1}.$$

Exercise 1.6. In each of the following equations, introduce the new variables indicated to reduce the equation to one of the forms in Exercise 4 above; then find the general solution of the equation.

- (a) $4u_{xx} - u_{yy} = 0 ; s = x + 2y, t = x - 2y.$
- (b) $u_{xx} - 6u_{xy} + 5u_{yy} = 0 ; s = 5x + y, t = x + y.$
- (c) $u_{xx} + 4u_{xy} + 4u_{yy} = 0 ; s = -2x + y, t = y.$
- (d) $u_{xx} - 2u_{xy} + u_{yy} = 0 ; s = x, t = x + y.$
- (e) $u_{xx} + 4u_{xy} = 0 ; s = 2x + iy, t = 2x - iy$ where $i = \sqrt{-1}.$
- (f) $u_{xx} - 4u_{xy} + 13u_{yy} = 0 ; s = (2 + 3i)x + y, t = (2 - 3i)x + y.$

Exercise 1.7. Let the partial differential equation

$$\frac{\partial^2}{\partial x^2} u(x, y) - \frac{\partial^2}{\partial y^2} u(x, y) + 2 \frac{\partial}{\partial x} u(x, y) + u(x, y) = 0 \dots\dots\dots (E_1)$$

1. If $\xi = x - y, \eta = x + y$ and $v(\xi, \eta) = u(x, y)$, show that the equation (E₁) is equivalent to the following equation

$$2 \frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{v}{2} = 0 \dots\dots\dots (E_2)$$

2. To get the solution of the equation (E₂), we set the following change of the function

$$v(\xi, \eta) = \varphi(\xi, \eta) e^{-\frac{1}{2}(\xi+\eta)}.$$

a) Determine the equation satisfied by φ , then deduce the solution of (E_2) and then the solution of (E_1) .

Chapter 2

Partial differential equations of first order

This chapter is concerned with first-order, quasi-linear and linear partial differential equations and their solution by using the Lagrange method of characteristics and its generalizations.

2.1 Classification of first-order equations

The most general, first-order, partial differential equation in two independent variables x and y is of the form

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset \mathbb{R} \quad (2.1)$$

where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in \mathbb{R} .

Equation (2.1) is often written in terms of standard notation $p = u_x$ and $q = u_y$ so that (2.1) takes the form

$$F(x, y, u, p, q) = 0. \quad (2.2)$$

Similarly, the most general, first-order, partial differential equation in three independent variables x, y, z can be written as

$$F(x, y, z, u, u_x, u_y, u_z) = 0. \quad (2.3)$$

Equation (2.1) or (2.2) is called a ***quasi-linear partial differential equation*** if it is linear in first-partial derivatives of the unknown function $u(x, y)$.

So, the most general quasi-linear equation must be of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad (2.4)$$

where its coefficients a, b and c are functions of x, y and u .

Equation (2.4) is called a **semilinear partial differential equation** if its coefficients a and b are independent of u , and hence, the semilinear equation can be expressed in the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \quad (2.5)$$

Equation (2.1) is said to be linear if F is linear in each of the variables u, u_x and u_y , and the coefficients of these variables are functions only of the independent variables x and y . The most general, **first-order linear partial differential equation** has the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \quad (2.6)$$

where the coefficients a, b and c , in general, are functions of x and y and $d(x, y)$ is a given function. Unless stated otherwise, these functions are assumed to be continuously differentiable.

Equations of the form (2.6) are called **homogeneous** if $d(x, y) = 0$ or **nonhomogeneous** if $d(x, y) \neq 0$.

Obviously, linear equations are a special kind of the quasi-linear equation (2.4) if a, b are independent of u and c is a linear function in u . Similarly, semilinear equation (2.5) reduces to a linear equation if c is linear in u .

An equation which is not linear is often called a **nonlinear equation**. So, first-order equations are often classified as linear and nonlinear.

2.2 Linear first order equations

In this section we shall study linear first-order differential equations in two independent variables x and y . The general form of such an equation is

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \quad (2.7)$$

where the coefficients a, b, c and d are functions in some domain D of the xy -plane. We assume that the functions a, b, c and d are continuous and have continuous partial derivatives of the first

order in D , and that a and b are not both zero.

We shall describe here a method for finding the general solution of equation (2.7). The method is based on the possibility of transforming equation (2.7) into an equation of the form

$$w_\xi + s(\xi, \eta)w = t(\xi, \eta). \quad (2.8)$$

where ξ and η are new independent variables. Equation (2.8) can now be regarded as an ordinary differential equation with ξ as the independent variable and η as a parameter. Hence, its general solution can be found by the use of the standard formula pertaining to such an equation. The integration constant in the solution must, however, be replaced by an arbitrary function of η . The general solution of the original equation is then obtained by returning to the variables x, y .

In order to transform equation (2.7) into the form (2.8) we introduce new variables ξ and η by the equations

$$\xi = \phi(x, y) \quad \eta = \psi(x, y). \quad (2.9)$$

The functions ϕ and ψ , which will be subsequently determined, are assumed to be continuous and have continuous first-order partial derivatives in D such that the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \phi_x \psi_y - \phi_y \psi_x \neq 0. \quad (2.10)$$

in the neighborhood of some point in D . Writing $u(x, y) = w(\xi, \eta)$, we then have by the chain rule,

$$\begin{aligned} u_x &= w_\xi \phi_x + w_\eta \psi_x \\ u_y &= w_\xi \phi_y + w_\eta \psi_y \end{aligned}$$

and hence equation (2.7) is transformed into the equation

$$(a\phi_x + b\phi_y)w_\xi + (a\psi_x + b\psi_y)w_\eta + cw = d. \quad (2.11)$$

The coefficients a, b, c and d are now to be considered as functions of ξ and η . if we choose the function $\psi(x, y)$ in (2.9) such that

$$a\psi_x + b\psi_y = 0 \quad \text{or} \quad \frac{\psi_x}{\psi_y} = -\frac{b}{a} \quad (2.12)$$

then the term involving w_η in (2.11) drops out, and for any choice of ϕ satisfying the condition

(2.10). equation (2.11) reduces to the form (2.8). Thus, it remains for us to determine a function ψ that satisfies (2.12).

Suppose for the moment that such a function ψ exists with $\psi_y \neq 0$; set $\psi(x, y) = c$, where c is any constant. Then, on taking the total differential of $\psi = c$, we find

$$\psi_x dx + \psi_y dy = 0.$$

which implies that

$$\frac{dy}{dx} = -\frac{\psi_x}{\psi_y}.$$

Hence, if ψ satisfies equation (2.12). then $\psi = c$ must be an integral of the first-order ordinary differential equation

$$\frac{dy}{dx} = \frac{b}{a}. \quad (2.13)$$

Conversely, if $\psi = c$ with $\psi_y \neq 0$ is an integral of equation (2.13), then by reversing the foregoing argument it follows that $\eta = \psi(x, y)$ satisfies equation (2.12).

Definition 8. Equation (2.13) is often called the characteristic equation of the partial differential equation (2.7). It defines a one-parameter family of curves called the characteristic curves of equation (2.7).

These curves play an important role in the consideration of initial value problems for equation (2.7), as we shall soon see.

Thus, in order to reduce equation (2.11) to the form (2.8), we choose ψ such that $\psi = c$ is a characteristic curve of equation (2.7). The function ϕ may be chosen arbitrary, subject to the condition (2.10). We choose $\phi(x, y) = x$. Then condition (2.10) is fulfilled and equation (2.11) takes the form

$$w_\xi + \left(\frac{c}{a}\right)w = \left(\frac{d}{a}\right). \quad (2.14)$$

which is the desired from (2.8).

Definition 9. This equation (2.14) is called the canonical form of equation (2.7) in terms of the coordinates (ξ, η) . Generally, the canonical equation (2.14) can easily be integrated and the general solution of (2.7) can be obtained after replacing ξ and η by the original variables x and y .

To obtain the general solution of equation (2.14), we multiply both sides of the equation by the integrating factor

$$v(\xi, \eta) = \exp\left(\int \frac{c}{a} d\xi\right).$$

and observe that

$$\frac{\partial}{\partial \xi} \left(w \exp\left[\int \frac{c}{a} d\xi\right] \right) = \left(w_\xi + \frac{c}{a} w \right) v(\xi, \eta) = \frac{d}{a} v(\xi, \eta).$$

Integrating with respect to ξ , treating η as a parameter, we then find

$$w(\xi, \eta) = \frac{1}{v(\xi, \eta)} \left[\int \frac{d}{a} v(\xi, \eta) d\xi + f(\eta) \right]$$

where f is an arbitrary function of η .

Hence, returning to the original variables x, y , we obtain as the general solution of equation (2.7), we have

$$u(x, y) = \frac{1}{v[x, \psi(x, y)]} \left\{ W[x, \psi(x, y)] + f[\psi(x, y)] \right\} \quad (2.15)$$

where we have set

$$W(\xi, \eta) = \int \frac{d}{a} v(\xi, \eta) d\xi.$$

We note that the first term, $W(x, \psi)/v(x, \psi)$, in formula (2.15) is actually a particular solution of equation (2.7), and the second term $f(x, \psi)/v(x, \psi)$, is the general solution of the corresponding homogeneous equation

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \quad (2.16)$$

In many special cases, it may be easier to find a particular solution of (2.7) by some other technique. For instance, the method of undetermined coefficients used to determine particular solutions in ordinary differential equations may be employed under similar conditions.

Example 6. Find the general solution of equation

$$au_x + bu_y + cu = 0$$

where a, b, c are constants and $a \neq 0$.

Solution : We see that the characteristic equation

$$\frac{dy}{dx} = \frac{b}{a}$$

defines a one-parameter family of characteristic curves

$$bx - ay = \text{const}$$

which are straight lines. Introducing the new variables $\xi = x = \phi$, $\eta = bx - ay = \psi$, the given equation becomes

$$w_\xi + \frac{c}{a}w = 0.$$

The general solution of this equation is

$$w(\xi, \eta) = e^{-c\xi/a} f(\eta)$$

where f is an arbitrary differentiable function. Thus, the general solution of the given differential equation is

$$u(x, y) = e^{-cx/a} f(bx - ay).$$

If $a = 0$, then the differential equation reduces to $u_y + (c/b)u = 0$, whose general solution is readily obtained as

$$u(x, y) = e^{-cy/b} f(x).$$

Example 7. Obtain the general solution of the equation

$$xu_x - yu_y + u = x.$$

Solution: We consider the characteristic equation

$$\frac{dy}{dx} = -\frac{y}{x}.$$

by separation of variables we find that the characteristic curves are given by $xy = c$.

We therefore introduce $\xi = x = \phi$ and $\eta = xy = \psi$. The transformed equation is

$$w_\xi + \frac{1}{\xi}w = 1$$

whose general solution is

$$w(\xi, \eta) = \frac{\xi}{2} + \frac{1}{\xi}f(\eta).$$

Hence, the general solution of the original equation is

$$u(x, y) = \frac{x}{2} + \frac{1}{x}f(xy)$$

where f is an arbitrary differentiable function.

Theorem 2. (*The Cauchy problem for a first-order partial differential equation*). Suppose that C is a given curve in the (x, y) -plane with its parametric equations

$$x = x_0(t), \quad y = y_0(t),$$

where t belongs to an interval $I \subset \mathbb{R}$ and the derivatives $x'_0(t)$ and $y'_0(t)$ are piecewise continuous functions, such that $(x'_0)^2 + (y'_0)^2 \neq 0$. Also, suppose that $u = u_0(t)$ is a given function on the curve C . Then, there exists a solution $u = u(x, y)$ of the equation

$$F(x, y, u, u_x, u_y) = 0 \tag{2.17}$$

in a domain D of \mathbb{R}^2 containing the curve C for all $t \in I$, and the solution $u(x, y)$ satisfies the given initial data, that is,

$$u(x_0(t), y_0(t)) = u_0(t) \tag{2.18}$$

for all values of $t \in I$.

In short, the Cauchy problem is to determine a solution of equation (2.17) in a neighborhood of C , such that the solution $u = u(x, y)$ takes a prescribed value $u_0(t)$ on C . The curve C is called the *initial curve* of the problem, and $u_0(t)$ is called the *initial data*. Equation (2.18) is called the *initial condition* of the problem.

Remark 4. The solution of the Cauchy problem also deals with such questions as the conditions on the functions $F, x_0(t), y_0(t)$, and $u_0(t)$ under which a solution exists and is unique.

Example 8. Find the solution of the equation

$$2u_x - 3u_y + 2u = 2x$$

which assumes the value $u = x^2$ on the line $y = -x/2$.

Solution: by formula (2.15) the general solution of the differential equation is

$$u(x,y) = (x - 1) + e^{-x}f(3x + 2y)$$

Substituting $y = -x/2$ in the solution and setting $u = x^2$, we have

$$x^2 = x - 1 + e^{-x}f(2x)$$

or

$$f(2x) = (x^2 - x + 1)e^x$$

let $t = 2x$ so that $x = t/2$ then

$$f(t) = \left(\frac{t^2}{4} - \frac{t}{2} + 1 \right) e^{t/2}$$

thus. The solution of the problem is

$$\begin{aligned} u(x,y) &= x - 1 + e^{-x} \left[\frac{(3x + 2y)^2}{4} - \frac{(3x + 2y)}{2} + 1 \right] e^{(3x+2y)/2} \\ &= x - 1 + \left[\frac{(3x + 2y)^2}{4} - \frac{(3x + 2y)}{2} + 1 \right] e^{x+2y/2}. \end{aligned}$$

2.3 First-order quasi-linear partial differential equations

We begin this section with simultaneous DEs, which play an important role in the theory of partial DEs. Then we shall consider quasilinear partial DEs of the Lagrange type, and show that such equations can be solved rather easily provided we can find solutions of related simultaneous DEs.

2.3.1 Simultaneous DEs.

To solve simultaneous DEs of the form

$$\frac{dx}{P} = \frac{dx}{Q} = \frac{dx}{R} \tag{2.19}$$

where P , Q and R are functions of x, y, u , several different techniques are known. We shall present here only the following two methods.

The method of grouping.

If in $dx/P = du/R$ the variable y can be canceled or absent, leaving the equation in x and u only, then an integration of this equation gives

$$\phi(x, u) = c_1. \quad (2.20)$$

Again, if one variable, say, x , is absent or can be removed from $dy/Q = du/R$, then an integration of this equation leads to

$$\psi(x, u) = c_2. \quad (2.21)$$

The general solution of (2.19) is then the solutions (2.20) and (2.21) taken together.

Example 9. For the DE

$$\frac{dx}{u^2y} = \frac{dx}{u^2x} = \frac{dx}{y^2x} \quad (2.22)$$

we take first two fractions and cancel out u^2 , to get $dx/y = dy/x$ or $xdxdydy = 0$, which can be integrated to obtain

$$x^2 - y^2 = c_1$$

Again, we take second and third fractions and cancel out x , to have $dy/u^2 = du/y^2$ or $y^2dyu^2du = 0$, which on integration yields

$$y^3 - u^3 = c_2$$

Equations $x^2 - y^2 = c_1$ and $y^3 - u^3 = c_2$ taken together gives the solution of (2.22).

The method of multipliers.

By a proper choice of multipliers ℓ, n, m which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dx}{Q} = \frac{dx}{R} = \frac{\ell dx + mdy + ndu}{\ell P + mQ + nR} \quad (2.23)$$

so that $\ell P + mQ + nR = 0$. Then $\ell dx + mdy + ndu = 0$ and this can be solved to get the integral

$$\phi(x, y, u) = c_1. \quad (2.24)$$

Again we search for another set of multipliers λ, μ, ν

$$\frac{dx}{P} = \frac{dx}{Q} = \frac{dx}{R} = \frac{\lambda dx + \mu dy + \nu du}{\lambda P + \mu Q + \nu R} \quad (2.25)$$

so that $\lambda P + \mu Q + \nu R = 0$. Then $\lambda dx + \mu dy + \nu du = 0$ and this can be solved to get the integral

$$\psi(x, y, u) = c_2. \quad (2.26)$$

These two integrals (2.24) and (2.26) taken together give the required solution of (2.19).

Example 10. Find the solution of the following the DE

$$\frac{dx}{x(u^2 - y^2)} = \frac{dx}{-y(u^2 + x^2)} = \frac{dx}{u(x^2 + y^2)}. \quad (2.27)$$

Solution. First, let the multipliers $\ell = x, n = y, m = u$ so that each fraction is the same as

$$\frac{xdx + ydy + udu}{x^2(u^2 - y^2) - y^2(u^2 + x^2) + u^2(x^2 + y^2)} = \frac{xdx + ydy + udu}{0} \quad (2.28)$$

and hence $xdx + ydy + udu = 0$ which on integration gives

$$x^2 + y^2 + u^2 = c_1.$$

Again using the multipliers $1/x, 1/y, 1/u$ we obtain

$$\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{u}du = 0$$

which can be solved to get the integral

$$yu = c_2x.$$

Hence, $\phi(x, y, u) = x^2 + y^2 + u^2$ and $\psi(x, y, u) = \frac{yu}{x}$ are solution of (2.28).

2.3.2 Geometrical interpretation of a first-order equation

We consider now the case of the quasi-linear equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \quad (2.29)$$

A solution of (2.29) defines an integral surface $\mathcal{S} : u(x, y)$ is the Euclidean (x, y, u) space. The normal to this surface at each point $P(x, y, u)$ is the vector $\vec{n}_P(u_x, u_y, -1)$ and let \vec{v}_P be the vector $(a(x, y, u), b(x, y, u), c(x, y, u))$.

Then the equation (2.29) can be interpreted as the condition that each point P of the integral

surface \mathcal{S} the vector \vec{v}_P is tangent to the surface \mathcal{S} .

Suppose that $P \in \Omega$, where Ω is a domain in the (x, y, u) space and consider the vector field $V = \{\vec{v}_P, P \in \Omega\}$. We define as **characteristic curves**

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t), & t \in [a, b] \\ u = u(t) \end{cases}$$

the integral curves in Ω of the **characteristic system**

$$\begin{cases} \frac{dx}{dt} = a(x, y, u), \\ \frac{dy}{dt} = b(x, y, u), \\ \frac{du}{dt} = c(x, y, u). \end{cases}$$

The last system can be rewritten shortly as

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)},$$

which is an autonomous system of ODEs.

Assuming a, b and c to be of class $C^1(\mathbb{R})$ by the existence and uniqueness theorem for ODEs it follows that through each point $P_0(x_0, y_0, u_0) \in \Omega$ passes exactly one characteristic curve Γ_0 .

Definition 10. *The system of ordinary differential equations of the characteristic curve is given by*

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)},$$

*are called the **characteristic equations** of the quasi-linear equation (2.29).*

2.3.3 Method of characteristics and general solutions

We can use the geometrical interpretation of first-order, partial differential equations and the properties of characteristic curves to develop a method for finding the general solution of quasi-linear equations. This is usually referred to as the *method of characteristics* due to Lagrange.

Suppose that for $P(x, y, u) \in \Omega$, $\vec{V}(a, b, u) \neq (0, 0, 0)$. The characteristic curve

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t), & t \in [a, b] \\ u = u(t) \end{cases}$$

can be represented as the intersection of two surfaces

$$\begin{aligned} \Gamma &= \mathcal{S}_1 \cap \mathcal{S}_2 \\ \mathcal{S}_1 : \phi(x, y, u) &= c_1 \\ \mathcal{S}_2 : \psi(x, y, u) &= c_2 \end{aligned} \tag{2.30}$$

for which the normal vectors $\vec{n}_1(\phi_x, \phi_y, \phi_u)$ and $\vec{n}_2(\psi_x, \psi_y, \psi_u)$ are linearly independent at each point P , which means that

$$\text{rank} \begin{bmatrix} \phi_x & \phi_y & \phi_u \\ \psi_x & \psi_y & \psi_u \end{bmatrix} = 2. \tag{2.31}$$

Definition 11. A continuously differentiable function $\phi(x, y, u)$ is said to be a **first integral** of (2.33) if it is a constant on characteristic curves.

Definition 12. The first integrals $\phi(x, y, u)$ and $\psi(x, y, u)$ of (2.33) are functionally independent if (2.31) is fulfilled.

Suppose $\phi(x, y, u)$ and $\psi(x, y, u)$ are functionally independent first integrals and (2.30) holds. From

$$\begin{aligned} \phi(x(t), y(t), u(t)) &= c_1 \\ \psi(x(t), y(t), u(t)) &= c_2 \end{aligned}$$

it follows

$$\begin{aligned} \phi_x \dot{x} + \phi_y \dot{y} + \phi_u \dot{u} &= 0 \\ \psi_x \dot{x} + \psi_y \dot{y} + \psi_u \dot{u} &= 0 \end{aligned}$$

where $\dot{x} = \frac{dx}{dt}$ and

$$\begin{aligned} \phi_x a + \phi_y b + \phi_u c &= 0 \\ \psi_x a + \psi_y b + \psi_u c &= 0. \end{aligned}$$

From (2.31) it follows that ϕ and ψ are functionally independent first integrals iff

$$\frac{dx}{\begin{vmatrix} \phi_y & \phi_u \\ \psi_y & \psi_u \end{vmatrix}} = \frac{dy}{\begin{vmatrix} \phi_u & \phi_x \\ \psi_u & \psi_x \end{vmatrix}} = \frac{du}{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}}, \quad (2.32)$$

which geometrically means that the vector $\vec{n}_1 \times \vec{n}_2$ is a tangent vector to Γ at P .

Now, the method of solution of quasi-linear equations can be described by the following result.

Theorem 3. *The general solution of a first-order, quasi-linear partial differential equation*

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad (2.33)$$

is

$$f(\phi, \psi) = 0$$

where f is an arbitrary function of $\phi(x, y, u)$ and $\psi(x, y, u)$, and $\phi = \text{constant} = c_1$ and $\psi = \text{constant} = c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}.$$

The solution curves defined by $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are called the families of characteristic curves of equation (2.33).

Proof. Let $u = u(x, y)$ be a function for which

$$F(\phi(x, y, u(x, y)), \psi(x, y, u(x, y))) = 0 \quad (2.34)$$

Differentiating (2.34) with respect to x, y , we have

$$\begin{aligned} F_\phi(\phi_x + \phi_u u_x) + F_\psi(\psi_x + \psi_u u_x) &= 0, \\ F_\phi(\phi_y + \phi_u u_y) + F_\psi(\psi_y + \psi_u u_y) &= 0. \end{aligned}$$

Assuming $(F_\phi, F_\psi) \neq (0, 0)$ it follows

$$\begin{vmatrix} \phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y \end{vmatrix} = 0$$

or

$$(\phi_u \psi_y - \phi_y \psi_u)u_x + (\phi_x \psi_u - \phi_u \psi_x)u_y = \phi_y \psi_x - \phi_x \psi_y. \quad (2.35)$$

From (2.35) and (2.34) it follows $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$.

Conversely let $u = u(x, y)$ be a solution of (2.29), $\phi(x, y, u)$ and $\psi(x, y, u)$ be functionally independent first integrals of (2.29). Then, by (2.32), it follows (2.35). \square

Although our above discussion is only for two independent variables, it can be extended rather easily to the case of n independent variables. In fact, analytic methods can be used to prove the following general theorem.

Theorem 4. *If $\phi_i(x_1, x_2, \dots, x_n, u) = c_i$, $1 \leq i \leq n$ are independent solutions of the equations*

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{du}{R}$$

then the relation $\Psi(\phi_1, \phi_2, \dots, \phi_n) = 0$, where the function Ψ is arbitrary, is a general solution of the linear partial DE

$$P_1 u_{x_1} + P_2 u_{x_2} + \dots + P_n u_{x_n} = R.$$

Example 11. *Find the solution of the following equation*

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0.$$

Solution. The associated characteristic system is given by:

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{du}{0}$$

The first integrals are given by:

$$\begin{aligned} \Phi_1(x, y, u) &= u, \\ \Phi_2(x, y, u) &= x^2 + y^2. \end{aligned}$$

Thus, the characteristic curves associated with this equation are defined as the intersection of two surfaces in \mathbb{R}^3 :

$$\mathcal{S}_1 = \{(x, y, z) : z = c_1, c_1 \in \mathbb{R}\}$$

and

$$\mathcal{S}_2 = \{(x, y, z) : x^2 + y^2 = c_2, c_2 \in \mathbb{R}\}.$$

Therefore, the characteristic curves are given by:

$$\Gamma = \mathcal{S}_1 \cap \mathcal{S}_2 = \{(x, y, u) : u = c \text{ and } x^2 + y^2 = c, c \in \mathbb{R}\}.$$

All solutions are thus implicitly defined by:

$$(x, y) \mapsto F(x^2 + y^2, u(x, y)) = \text{Constant}.$$

According to the implicit function theorem, this yields:

$$u(x, y) = g(x^2 + y^2),$$

for some C^1 function g .

Theorem 5. (*The Cauchy problem for a quasi-linear equation*). Suppose that $x_0(t)$, $y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in a closed interval, $0 \leq t \leq 1$, and that a, b , and c are functions of x, y , and u with continuous first-order partial derivatives with respect to their arguments in some domain D of (x, y, u) -space containing the initial curve

$$\Gamma : x = x_0(t), \quad y = y_0(t), \quad u = u_0(t), \quad (2.36)$$

where $0 \leq t \leq 1$ and satisfying the condition

$$y'_0(t)a(x_0(t), y_0(t), u_0(t)) - x'_0(t)b(x_0(t), y_0(t), u_0(t)) \neq 0. \quad (2.37)$$

Then there exists a unique solution $u = u(x, y)$ of the quasi-linear equation (2.33) in the neighborhood of $C : x = x_0(t)$, $y = y_0(t)$, and the solution satisfies the initial condition

$$u_0(t) = u(x_0(t), y_0(t)), \quad 0 \leq t \leq 1.$$

Example 12. Find the solution of the equation

$$u(x + y)u_x + u(x - y)u_y = x^2 + y^2, \quad (2.38)$$

with the Cauchy data $u = 0$ on $y = 2x$.

Solution. The characteristic equations are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2}.$$

First, let the multipliers $\ell = y, n = x, m = -u$ so that each fraction is the same as

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2} = \frac{ydx + xdy - udu}{0} \quad (2.39)$$

and hence $ydx + xdy - udu = 0$ which on integration gives

$$2xy - \frac{1}{2}u^2 = c_1.$$

where c_1 is a positive constant. Again using the multipliers x, y, u we obtain

$$xdx - ydy - udu = 0$$

which can be solved to get the integral

$$x^2 - y^2 - u^2 = c_2.$$

where c_2 is a positive constant. Hence, the general solution is

$$f(\phi(x, y, u), \psi(x, y, u)) = 0$$

where $\phi(x, y, u) = 2xy - \frac{1}{2}u^2$ and $\psi(x, y, u) = x^2 - y^2 - u^2$ and f is an arbitrary function.

Now, using the Cauchy data $u = 0$ on $y = 2x$, we obtain $4c_1 = 3c_2$. Therefore

$$4(2xy - \frac{1}{2}u^2) = 3(x^2 - y^2 - u^2).$$

Thus, the solution of equation (2.38) is given by

$$u^2(x, y) = 3x^2 - 3y^2 - 8xy.$$

2.4 Fully-nonlinear first-order equations

We shall now explain a general method for finding the solutions of first-order nonlinear partial DEs which is due to Charpit.

2.4.1 Charpit's method.

Consider the equation

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset \mathbb{R} \quad (2.40)$$

where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in \mathbb{R} .

Equation (2.1) is often written in terms of standard notation $p = u_x$ and $q = u_y$ so that (2.1) takes the form

$$F(x, y, u, p, q) = 0. \quad (2.41)$$

Since u depends on x and y , we have

$$du = u_x dx + u_y dy = p dx + q dy. \quad (2.42)$$

If we can find another relation involving x, y, u, p, q such as

$$\varphi(x, y, u, p, q) = 0 \quad (2.43)$$

then we can solve (2.41) and (2.43) for p and q and substitute in (2.42). This will give the solution provided the resulting equation is integrable.

To determine φ , we differentiate (2.41) and (2.43) with respect to x and y , to obtain

$$\begin{cases} F_x + F_u p + F_p p_x + F_q q_x = 0, \\ \varphi_x + \varphi_u p + \varphi_p p_x + \varphi_q q_x = 0, \\ F_y + F_u q + F_p p_y + F_q q_y = 0, \\ \varphi_y + \varphi_u q + \varphi_p p_y + \varphi_q q_y = 0. \end{cases} \quad (2.44)$$

Eliminating p_x between the first equation and the second equation of (2.44), we get

$$(F_x \varphi_p - \varphi_x F_p) + (F_u \varphi_p - \varphi_u F_p) p + (F_q \varphi_p - \varphi_q F_p) q_x = 0 \quad (2.45)$$

Also eliminating q_y between the third equation and the fourth equation of (2.44), we obtain

$$(F_y\varphi_p - \varphi_y F_q) + (F_u\varphi_q - \varphi_u F_q)q + (F_p\varphi_q - \varphi_p F_q)p_y = 0. \quad (2.46)$$

Adding (2.45) and (2.46) and using $q_x = u_{xy} = p_y$, we find that the last terms in both cancel and the other terms, on rearrangement, give

$$(-F_x)\varphi_x + (-F_q)\varphi_y + (-pF_p - qF_q)\varphi_u + (F_x + pF_u)\varphi_p + (F_y + qF_u)\varphi_q = 0.$$

This is Lagrange's DE with x, y, u, p, q as independent variables and φ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{du}{-pF_p - qF_q} = \frac{dp}{F_x + pF_u} = \frac{dq}{F_y + qF_u} = \frac{d\varphi}{0}.$$

An integral of these equations involving p or q or both, can be taken as the required relation (2.43).

Working rule while using Charpit's method:

Step 1: Transfer all the terms of the given equation to L. H. S. and denote entire expression by F .

Step 2: Write down the Charpit's equations, namely

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{du}{-pF_p - qF_q} = \frac{dp}{F_x + pF_u} = \frac{dq}{F_y + qF_u} = \frac{d\varphi}{0}. \quad (2.47)$$

Step 3: Using the value of F in the **Step 1**, write down the values of F_p, F_q, \dots , etc occurring in **Step 2** and put these in the Charpit's equations.

Step 4: After simplifying the **Step 3**, select two proper fractions so that the resulting integral may come out to be simplest relation involving at least one of p and q .

Step 5: The simple relation of **Step 4** is solved along with the given equation to determine p and q . Put these values of p and q in the relation $du = pdx + qdy$ with on integration gives the general solution of the nonlinear equation.

Example 13. Find the general solution of $u_x^2 + yu_y - u = 0$

Solution. First we introduce $u_x = p$ and $u_y = q$. Then

$$F(x, y, u, p, q) = p^2 + yq - u = 0.$$

Next we identify,

$$F_x = 0, \quad F_y = q, \quad F_p = 2p, \quad F_q = y, \quad F_u = -1.$$

Then

$$-pF_p - qF_q = -2p^2 - qy,$$

$$F_x + pF_u = -p,$$

$$F_y + qF_u = 0.$$

The Charpit equations are then

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{du}{-2p^2 - qy} = \frac{dp}{-p} = \frac{dq}{0}.$$

The first conclusion is that $q = c_1 = \text{constant}$. So, from the partial differential equation we have $u = p^2 + c_1y$.

Since $du = pdx + qdy$, then $du = pdx + c_1dy$ and therefore $du - c_1dy = pdx$ but $p = \sqrt{u - c_1y}$, hence $d(u - c_1y) = \sqrt{u - c_1y}dx$ which on integration gives

$$\int \frac{d(u - c_1y)}{\sqrt{u - c_1y}} = \int dx \Rightarrow 2\sqrt{u - c_1y} = x + c_2$$

where c_2 is an arbitrary constant. Solving for u , we have

$$u(x, y) = \frac{1}{4}(x + c_2)^2 + c_1y.$$

2.5 Exercises

Exercise 2.1. Solve the following first-order linear partial DEs

$$(1) \quad u_x + u_y - u = 0;$$

$$(2) \quad 2u_x - 3u_y = x;$$

$$(3) \quad u_x - 2u_y + u = \sin x + y;$$

$$(4) \quad 3u_x - 4u_y + 2u = x^2y + 2e^x + 1;$$

$$(5) \quad u_x - u_y - 2u = e^{2x} \cos(3y).$$

Exercise 2.2. Solve the following simultaneous DEs:

$$(1) \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{nxy};$$

$$(2) \frac{dx}{mu - ny} = \frac{dy}{nx - lu} = \frac{du}{ly - mx};$$

$$(3) \frac{dx}{x^2 - yu} = \frac{dy}{y^2 - ux} = \frac{du}{u^2 - xy};$$

$$(4) \frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2 + y^2};$$

$$(5) \frac{dx}{x(y^2 - u^2)} = \frac{dy}{y(u^2 - x^2)} = \frac{du}{u(x^2 - y^2)};$$

$$(6) \frac{dx}{x^2 - y^2 - u^2} = \frac{dy}{2xy} = \frac{du}{2xu}.$$

Exercise 2.3. Solve the following first-order quasilinear partial DEs:

$$(1) (x^2 - y^2 - u^2)u_x + 2xyu_y = 2xu;$$

$$(2) (y - u)u_x + (x - y)u_y = u - x;$$

$$(3) (x^2 - yu)u_x + (y^2 - ux)u_y = u^2 - xy;$$

$$(4) y^2u_x - xyu_y = x(u - 2y)$$

$$(5) x^2u_x + y^2u_y = (x + y)u.$$

Exercise 2.4. Solve the following first-order quasilinear partial DEs:

$$(i) yz \frac{\partial u}{\partial x} + xz \frac{\partial u}{\partial y} - xy \frac{\partial u}{\partial z} = 0$$

$$(ii) -y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + (1 + z^2) \frac{\partial u}{\partial z} = 3zu,$$

$$(iii) yz \frac{\partial u}{\partial x} + xz \frac{\partial u}{\partial y} + xy \frac{\partial u}{\partial z} + xyz = 0,$$

$$(iv) x(cu - by) \frac{\partial u}{\partial x} + y(ax - cu) \frac{\partial u}{\partial y} = u(by - ax).$$

Exercise 2.5. Solve the following first-order nonlinear partial DEs:

$$(1) u = p^2x + q^2y;$$

$$(2) u^2 = xypq;$$

$$(3) 1 + p^2 = qu;$$

$$(4) pxy + pq + qy = yu$$

$$(5) u^2(p^2 + q^2) = x^2 + y^2$$

where $p = u_x$ and $q = u_y$.

Chapter 3

Partial differential equations of second order

Many physical problems related to fluid mechanics, heat transfer, dynamics, and electromagnetism are described by second-order partial differential equations (PDEs). Therefore, studying their solution methods is essential for solving real-world problems.

3.1 Classification and canonical forms of equations in two independent variables

Here we shall be concerned with second-order equations in the dependent variable u and the independent variables x and y . The general form of a linear second-order equation in two independent variables is

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + D(x, y) \frac{\partial u}{\partial x} + E(x, y) \frac{\partial u}{\partial y} + F(x, y)u = G(x, y) \quad (\mathcal{E})$$

where $A, B, C, D, E, F, G \in C^2(\Omega)$, $\Omega \in \mathbb{R}^2$ and $A^2 + B^2 + C^2 \neq 0$. We shall assume that the function u and the coefficients are twice continuously differentiable in some domain in \mathbb{R}^2 .

The classification of partial differential equations is suggested by the classification of *the quadratic equation of conic sections in analytic geometry*. The equation

$$Ax^2 + Axy + Cy^2 + Dx + Ey + F = 0,$$

represents hyperbola, parabola, or ellipse accordingly as $B^2 - 4AC$ is positive, zero, or negative.

The classification of second-order equations is based upon the possibility of reducing equation (\mathcal{E}) by coordinate transformation to canonical or standard form at a point. An equation is said to be **hyperbolic**, **parabolic**, or **elliptic** at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0)$$

is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic in a domain. Thus, we define:

Definition 13. Let $\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y)$, then we say that the equation (\mathcal{E}) at a point $P(x, y) \in \Omega$ is:

1. **hyperbolic**, if $\Delta(x, y) > 0$,
2. **parabolic**, if $\Delta(x, y) = 0$,
3. **elliptic**, if $\Delta(x, y) < 0$.

Example 14. The wave equation

$$\frac{\partial^2 u}{\partial t^2} - c \frac{\partial^2 u}{\partial x^2} = 0$$

is a hyperbolic equation on the domain $D = \mathbb{R}_+ \times \mathbb{R}$ because

$$\Delta(t, x) = b^2(t, x) - 4a(t, x)c(t, x) = 4c^2 > 0.$$

Example 15. Consider the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin(x + y),$$

we compute the discriminant $\Delta(x, y) = -3x^2y^2$. Therefore, if $x = 0$ or $y = 0$, Δ vanishes. So, on the domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\},$$

this PDE is parabolic, and it is elliptic on $\mathbb{R}^2 \setminus D$.

To transform the equation (\mathcal{E}) into a canonical form, we need to perform a change of independent variables.

3.2 Change of variables

To transform equation (\mathcal{E}) to a canonical form we make a change of independent variables. Let the new variables be

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

Assuming that ξ and η are twice continuously differentiable and that the Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad (3.1)$$

in the region under consideration, then x and y can be determined uniquely from the system (3.1).

Definition 14. *The transformation $(\xi, \eta) = (\xi(x, y), \eta(x, y))$ is called a change of coordinates (or a nonsingular transformation) if the Jacobian $J := \xi_x \eta_y - \eta_x \xi_y$ of the transformation does not vanish at any point (x, y) .*

Proposition 1. *The type of a linear second-order PDE in two variables is invariant under a change of coordinates. In other words, the type of the equation is an intrinsic property of the equation and is independent of the particular coordinate system used.*

Proof. Let the equation (\mathcal{E}) and let $(\xi, \eta) = (\xi(x, y), \eta(x, y))$ be a nonsingular transformation. Write $u(x(\xi, \eta), y(\xi, \eta)) = v(\xi, \eta)$. We claim that v is a solution of a second-order equation of the same type. Using the chain rule one finds that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y}. \end{aligned}$$

Based on the two equations above, and again using the chain rule, we express the second-order derivatives as follows:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2 v}{\partial \xi^2} + 2 \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) \frac{\partial^2 v}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 v}{\partial \eta^2} \\ &\quad + \left(\frac{\partial^2 \xi}{\partial x^2} \right) \frac{\partial v}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x^2} \right) \frac{\partial v}{\partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) \frac{\partial^2 v}{\partial \xi^2} + \left[\left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} \right) \right] \frac{\partial^2 v}{\partial \xi \partial \eta} \\ &\quad + \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) \frac{\partial^2 v}{\partial \eta^2} + \left(\frac{\partial^2 \xi}{\partial x \partial y} \right) \frac{\partial v}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial x \partial y} \right) \frac{\partial v}{\partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \left(\frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2 v}{\partial \xi^2} + 2 \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right) \frac{\partial^2 v}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 v}{\partial \eta^2} \\ &\quad + \left(\frac{\partial^2 \xi}{\partial y^2} \right) \frac{\partial v}{\partial \xi} + \left(\frac{\partial^2 \eta}{\partial y^2} \right) \frac{\partial v}{\partial \eta}. \end{aligned}$$

By substituting these expressions into the equation (\mathcal{E}) , the following equation is obtained:

$$A^* \frac{\partial^2 v}{\partial \xi^2} + B^* \frac{\partial^2 v}{\partial \xi \partial \eta} + C^* \frac{\partial^2 v}{\partial \eta^2} + D^* \frac{\partial v}{\partial \xi} + E^* \frac{\partial v}{\partial \eta} + F^* v = G^* \quad (\mathcal{E}^*)$$

where:

$$\begin{aligned} A^* &= A \left(\frac{\partial \xi}{\partial x} \right)^2 + B \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) + C \left(\frac{\partial \xi}{\partial y} \right)^2, \\ B^* &= 2A \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial x} \right) + B \left[\left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} \right) \right] + 2C \left(\frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial y} \right), \\ C^* &= A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + C \left(\frac{\partial \eta}{\partial y} \right)^2, \\ D^* &= A \left(\frac{\partial^2 \xi}{\partial x^2} \right) + B \left(\frac{\partial^2 \xi}{\partial x \partial y} \right) + C \left(\frac{\partial^2 \xi}{\partial y^2} \right) + D \left(\frac{\partial \xi}{\partial x} \right) + E \left(\frac{\partial \xi}{\partial y} \right) \\ E^* &= A \left(\frac{\partial^2 \eta}{\partial x^2} \right) + B \left(\frac{\partial^2 \eta}{\partial x \partial y} \right) + C \left(\frac{\partial^2 \eta}{\partial y^2} \right) + D \left(\frac{\partial \eta}{\partial x} \right) + E \left(\frac{\partial \eta}{\partial y} \right) \\ F^* &= F, \quad G^* = G. \end{aligned} \tag{3.2}$$

The resulting equation (\mathcal{E}^*) is in the same form as the original equation (\mathcal{E}) under the general transformation $(\xi, \eta) = (\xi(x, y), \eta(x, y))$. The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This can be seen from the fact that the sign of the discriminant does not alter under the transformation, that is,

$$B^{*2}(\xi, \eta) - 4A^*(\xi, \eta)C^*(\xi, \eta) = J^2 (B^2(x, y) - 4A(x, y)C(x, y))$$

which can be easily verified. Since $J \neq 0$, the two equations (\mathcal{E}) and (\mathcal{E}^*) are of the same type. \square

Remark 5. *The classification of equation (\mathcal{E}) depends on the coefficients $A(x, y)$, $B(x, y)$ and*

$C(x, y)$ at a given point (x, y) . We shall, therefore, rewrite equation (\mathcal{E}) as

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = H(x, y, u, u_x, u_y) \quad (\mathcal{E}),$$

and equation (\mathcal{E}^*) as

$$A^*(\xi, \eta) \frac{\partial^2 v}{\partial \xi^2} + B^*(\xi, \eta) \frac{\partial^2 v}{\partial \xi \partial \eta} + C^*(\xi, \eta) \frac{\partial^2 v}{\partial \eta^2} = H^*(\xi, \eta, v, v_\xi, v_\eta) \quad (\mathcal{E}^*),$$

3.3 Canonical forms

Writing a second-order PDE in its canonical form allows us to reduce it to a simpler expression. This process eliminates certain second-order derivatives and helps in finding solutions. The canonical form of a second-order PDE depends on the type of the equation. For this purpose, We suppose first that none of A, B, C is zero. Let ξ and η be new variables such that the coefficients A^* and C^* in equation (4.1.11) vanish. Thus, from (3.2), we have

$$\begin{aligned} A^* &= A \left(\frac{\partial \xi}{\partial x} \right)^2 + B \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) + C \left(\frac{\partial \xi}{\partial y} \right)^2 = 0, \\ C^* &= A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + C \left(\frac{\partial \eta}{\partial y} \right)^2 = 0. \end{aligned}$$

These two equations are of the same type and hence we may write them in the form

$$A \left(\frac{\partial \zeta}{\partial x} \right)^2 + B \left(\frac{\partial \zeta}{\partial x} \right) \left(\frac{\partial \zeta}{\partial y} \right) + C \left(\frac{\partial \zeta}{\partial y} \right)^2 = 0, \quad (3.3)$$

in which ζ stand for either of the functions ξ or η . Dividing through by ζ_y^2 equation (3.3) becomes

$$A \left(\frac{\zeta_x}{\zeta_y} \right)^2 + B \left(\frac{\zeta_x}{\zeta_y} \right) + C = 0. \quad (3.4)$$

Along the curve $\zeta = \text{constant}$, we have

$$d\zeta = \zeta_x dx + \zeta_y dy.$$

Thus

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$$

and therefore, equation (3.4) may be written in the form

$$A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C = 0. \quad (3.5)$$

the roots of which are

$$\begin{aligned} \frac{dy}{dx} &= \frac{B + \sqrt{B^2 - 4AC}}{2A} \\ \frac{dy}{dx} &= \frac{B - \sqrt{B^2 - 4AC}}{2A} \end{aligned} \quad (3.6)$$

Definition 15. The equations (3.6), which are known as **the characteristic equations**, are ordinary differential equations for families of curves in the xy -plane along which $\xi = \text{constant}$ and $\eta = \text{constant}$. The integrals of equations (3.6) are called **the characteristic curves**.

Remark 6. Since the equations (3.6), are first-order ordinary differential equations, the solutions may be written as

$$\psi_1(x, y) = c_1$$

$$\psi_2(x, y) = c_2$$

where c_1 and c_2 are constants. Hence the transformations

$$\xi = \psi_1(x, y), \quad \eta = \psi_2(x, y)$$

will transform equation (\mathcal{E}) to a canonical form.

3.3.1 Canonical form of hyperbolic equations

Theorem 6. Suppose that (\mathcal{E}) is hyperbolic in a domain D . There exists a coordinate system (ξ, η) in which the equation has the canonical form

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = G \left(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta} \right)$$

where $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ and G is a function which depends on (\mathcal{E}).

Proof. Without loss of generality, we may assume that $A(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ such that

$$\begin{aligned} A^*(\xi, \eta) &= A \left(\frac{\partial \xi}{\partial x} \right)^2 + B \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) + C \left(\frac{\partial \xi}{\partial y} \right)^2 = 0, \\ C^*(\xi, \eta) &= A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + C \left(\frac{\partial \eta}{\partial y} \right)^2 = 0. \end{aligned}$$

The equation that was obtained for the function η is actually the same equation as for ξ ; therefore, we need to solve only one equation. It is a first-order equation that is not quasilinear; but as a quadratic form in ξ it is possible to write it as a product of two linear terms

$$\frac{1}{A} \left[A \left(\frac{\partial \xi}{\partial x} \right) + \frac{B + \sqrt{B^2 - 4AC}}{2A} \left(\frac{\partial \xi}{\partial y} \right) \right] \left[A \left(\frac{\partial \xi}{\partial x} \right) + \frac{B - \sqrt{B^2 - 4AC}}{2A} \left(\frac{\partial \xi}{\partial y} \right) \right] = 0.$$

Therefore, we need to solve the following linear equations:

$$\begin{aligned} A \left(\frac{\partial \xi}{\partial x} \right) + \frac{B + \sqrt{B^2 - 4AC}}{2A} \left(\frac{\partial \xi}{\partial y} \right) &= 0, \\ A \left(\frac{\partial \xi}{\partial x} \right) + \frac{B - \sqrt{B^2 - 4AC}}{2A} \left(\frac{\partial \xi}{\partial y} \right) &= 0. \end{aligned} \tag{3.7}$$

In order to obtain a nonsingular transformation $(\xi(x, y), \eta = \eta(x, y))$ we choose ξ to be a solution of the first equation of (3.7) and η to be a solution of the second equation of (3.7).

$$\frac{dx}{dt} = A, \quad \frac{dy}{dt} = B + \sqrt{B^2 - 4AC}, \quad \frac{d\xi}{dt} = 0.$$

Therefore, ξ is constant on each characteristic. The characteristics are solutions of the equation

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}.$$

The function η is constant on the characteristic determined by

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}.$$

□

Example 16. Find the canonical for the following equation:

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

Solution. The discriminant is $\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y) = 4x^2y^2$.

Therefore, if $x = 0$ or $y = 0$, the PDE is parabolic at the point (x, y) ; otherwise, it is hyperbolic.

Let us consider a domain D in which the PDE is hyperbolic at every point. In this domain,

the characteristic equations are:

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{4x^2y^2}}{2y^2} = \pm \frac{x}{y}$$

Using the method of separation of variables, we obtain: $\frac{1}{2}(x^2 + y^2) = C_1$ and $\frac{1}{2}(x^2 - y^2) = C_2$, where C_1 and C_2 are constants. The characteristic curves are:

$$\begin{cases} \varphi_1(x, y) = x^2 + y^2, \\ \varphi_2(x, y) = x^2 - y^2. \end{cases}$$

We define:

$$\begin{cases} \xi = x^2 + y^2, \\ \eta = x^2 - y^2. \end{cases}$$

and we set $u(x, y) = v(\xi, \eta)$. Using this coordinate transformation, we get:

$$\begin{cases} x^2 = \frac{\xi + \eta}{2}, \\ y^2 = \frac{\xi - \eta}{2}. \end{cases}$$

By the chain rule, we compute:

$$\begin{aligned} \frac{\partial u}{\partial x} &= -x \frac{\partial v}{\partial \xi} + x \frac{\partial v}{\partial \eta}, \\ \frac{\partial u}{\partial y} &= y \frac{\partial v}{\partial \xi} + y \frac{\partial v}{\partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(-x \frac{\partial v}{\partial \xi} + x \frac{\partial v}{\partial \eta} \right) \\ &= x^2 \frac{\partial^2 v}{\partial \xi^2} - 2x^2 \frac{\partial^2 v}{\partial \xi \partial \eta} + x^2 \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(y \frac{\partial v}{\partial \xi} + y \frac{\partial v}{\partial \eta} \right) \\ &= y^2 \frac{\partial^2 v}{\partial \xi^2} + 2y^2 \frac{\partial^2 v}{\partial \xi \partial \eta} + y^2 \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}. \end{aligned}$$

Finally, substituting into the original equation:

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = -4x^2 y^2 \frac{\partial^2 v}{\partial \xi \partial \eta} - (x^2 + y^2) \frac{\partial v}{\partial \xi} + (y^2 - x^2) \frac{\partial v}{\partial \eta} = 0,$$

which implies, after simplification:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{\eta}{2(\xi^2 - \eta^2)} \frac{\partial v}{\partial \xi} - \frac{\xi}{2(\xi^2 - \eta^2)} \frac{\partial v}{\partial \eta}.$$

This is the canonical form of the PDE at the points where it is hyperbolic.

3.3.2 Canonical form of parabolic equations

Theorem 7. *Suppose that (\mathcal{E}) is parabolic in a domain D . There exists a coordinate system (ξ, η) in which the equation has the canonical form*

$$\frac{\partial^2 v}{\partial \xi^2} = G\left(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta}\right).$$

where $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ and G is a function which depends on (\mathcal{E}) .

Proof. Since $B^2 - 4AC = 0$, we may assume that $A(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y), \eta = \eta(x, y)$ such that $B^*(\xi, \eta) = C^*(\xi, \eta) = 0$ for all $(x, y) \in D$. It is enough to make $C^* = 0$, since the parabolicity of the equation will then imply that $B^* = 0$.

Therefore, we need to find a function η that is a solution of the equation

$$C^*(\xi, \eta) = A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + C \left(\frac{\partial \eta}{\partial y} \right)^2 = \frac{1}{A} \left[A \left(\frac{\partial \eta}{\partial x} \right) + \frac{B}{2} \left(\frac{\partial \eta}{\partial y} \right) \right]^2 = 0.$$

From this it follows that η is a solution of the first-order linear equation:

$$A \left(\frac{\partial \eta}{\partial x} \right) + \frac{B}{2} \left(\frac{\partial \eta}{\partial y} \right) = 0.$$

Hence, the solution η is constant on each characteristic, i.e., on a curve that is a solution of the equation

$$\frac{dy}{dx} = \frac{B}{2A}.$$

Now, the only constraint on the second independent variable ξ , is that the Jacobian of the transformation should not vanish in D , and we may take any such function ξ . Note that a parabolic equation admits only one family of characteristics while for hyperbolic equations we have two

families. □

Example 17. Find the canonical form for the following equation:

$$\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

Solution. This equation is parabolic on \mathbb{R}^2 . The characteristic equation is:

$$\frac{dy}{dx} = 3,$$

whose solution is $3x - y = C$. The first characteristic curve is therefore:

$$\varphi_1(x, y) = 3x - y.$$

We then set $\xi = 3x - y$. To obtain a second characteristic coordinate η , there are many possible choices. For this example, we take $\eta(x, y) = x$. This function has continuous partial derivatives of order $m \leq 2$, and

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0,$$

for all points in \mathbb{R}^2 . Using this coordinate transformation:

$$\begin{cases} \xi = 3x - y, \\ \eta = x, \end{cases}$$

and setting $u(x, y) = v(\xi, \eta)$, we get:

$$\begin{cases} x = \eta, \\ y = 3\eta - \xi. \end{cases}$$

Using the chain rule, we obtain:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial \xi}, \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \right) = 9\frac{\partial^2 v}{\partial \xi^2} + 6\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2},$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial y} \left(3 \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \right) = -3 \frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \xi \partial \eta}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 v}{\partial \xi^2}.\end{aligned}$$

Finally,

$$\frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial^2 u}{\partial x \partial y} + 9 \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = \frac{\partial^2 v}{\partial \eta^2} - 5 \frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta},$$

which implies:

$$\frac{\partial^2 v}{\partial \eta^2} = 5 \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}.$$

3.3.3 Canonical form of elliptic equations

We should choose ξ and η such that $A = C$ and $B = 0$. Since $B^2 - 4AC < 0$ it follows that $A \neq 0$ and **the characteristic equations** of the equation (\mathcal{E}) given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{B + i\sqrt{-\Delta}}{2A} \\ \frac{dy}{dx} &= \frac{B - i\sqrt{-\Delta}}{2A}\end{aligned}\tag{3.8}$$

are ordinary differential equations of the complex form.

Let $\varphi(x, y) = \xi(x, y) + i\eta(x, y) = \text{constant}$, be the general solutions of the first equation of (3.8). Moreover, the solutions of the two equations are complex conjugates.

Theorem 8. *Suppose that (\mathcal{E}) is elliptic in a planar domain D . Assume further that the coefficients A, B, C are real analytic functions in D . Then there exists a coordinate system (ξ, η) in which the equation (\mathcal{E}) has the canonical form*

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = G \left(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta} \right),$$

where $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$.

Proof. Assume that $A(x, y) \neq 0$ for all $(x, y) \in \Omega$. We are looking for two function $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ that satisfy the equations

$$\begin{aligned}A^* &= A \left(\frac{\partial \xi}{\partial x} \right)^2 + B \left(\frac{\partial \xi}{\partial x} \right) \left(\frac{\partial \xi}{\partial y} \right) + C \left(\frac{\partial \xi}{\partial y} \right)^2 = C^* = A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \left(\frac{\partial \eta}{\partial x} \right) \left(\frac{\partial \eta}{\partial y} \right) + C \left(\frac{\partial \eta}{\partial y} \right)^2, \\ B^* &= 2A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B \left[\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right] + 2C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} = 0.\end{aligned}\tag{3.9}$$

This is a system of two nonlinear first-order equations. The main difficulty in the elliptic case is that (3.9) are coupled. In order to decouple these equations, we shall use the complex plane and the analyticity assumption. We may write the system (3.9) in the following form

$$\begin{aligned} A(\xi_x^2 - \eta_x^2) + B(\xi_x \xi_y - \eta_x \eta_y) + C(\xi_y^2 - \eta_y^2) &= 0 \\ 2A\xi_x i\eta_x + B(\xi_x i\eta_y + \xi_y i\eta_x) + 2C\xi_y i\eta_y &= 0, \end{aligned} \quad (3.10)$$

where $i^2 = -1$. By setting $\phi(x, y) = \xi + i\eta$, the system (3.10) is equivalent to the complex valued equation

$$A \left(\frac{\partial \phi}{\partial x} \right)^2 + B \left(\frac{\partial \phi}{\partial x} \right) \left(\frac{\partial \phi}{\partial y} \right) + C \left(\frac{\partial \phi}{\partial y} \right)^2 = 0.$$

Surprisingly, we have arrived at the same equation as in the hyperbolic case. But in the elliptic case the equation does not admit any real solution, or, in other words, elliptic equations do not have characteristics. Consequently, the characteristics are complex, and we must solve:

$$A \frac{\partial \phi}{\partial x} + \left(B \pm i\sqrt{4AC - B^2} \right) \frac{\partial \phi}{\partial y} = 0.$$

The associated characteristic system is given by

$$\frac{dy}{dx} = \frac{B \pm i\sqrt{B^2 - 4AC}}{2A}.$$

By setting

$$\begin{cases} \xi(x, y) = \operatorname{Re} \varphi(x, y), \\ \eta(x, y) = \operatorname{Im} \varphi(x, y), \end{cases} \quad (3.11)$$

the equation (\mathcal{E}) takes the following canonical form

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = G \left(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta} \right)$$

which is the canonical form of the elliptic equation. □

Example 18. Find the canonical form for the Tricomi equation

$$\frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial y^2} = 0, \quad x > 0.$$

Solution. We have

$$B^2(x, y) - 4A(x, y)C(x, y) = -4x = 4i^2x.$$

Since $x > 0$, this PDE is elliptic on the domain D . At these points, the characteristic equations are

$$\frac{dy}{dx} = \pm i\sqrt{x}.$$

Using the method of separation of variables, we obtain

$$\frac{3}{2}y \pm ix^{\frac{3}{2}} = C,$$

where C is a constant. We set

$$\begin{cases} \xi = \frac{3}{2}y, \\ \eta = -x^{\frac{3}{2}}. \end{cases}$$

and $u(x, y) = v(\xi, \eta)$. Using the chain rule, we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{3}{2}x^{\frac{1}{2}} \frac{\partial v}{\partial \eta}, \\ \frac{\partial u}{\partial y} &= \frac{3}{2} \frac{\partial v}{\partial \xi}, \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(-\frac{3}{2}x^{\frac{1}{2}} \frac{\partial v}{\partial \eta} \right) = \frac{9}{4}x \frac{\partial^2 v}{\partial \eta^2} - \frac{3}{4}x^{-\frac{1}{2}} \frac{\partial v}{\partial \eta},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{3}{2} \frac{\partial v}{\partial \xi} \right) = \frac{9}{4} \frac{\partial^2 v}{\partial \xi^2},$$

and finally

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial y^2} &= \frac{9}{4}x \frac{\partial^2 v}{\partial \eta^2} - \frac{3}{4}x^{-\frac{1}{2}} \frac{\partial v}{\partial \eta} + \frac{9}{4}x \frac{\partial^2 v}{\partial \xi^2} \\ &= \frac{9}{4} \left(\frac{\partial^2 v}{\partial \eta^2} + \frac{1}{3\eta} \frac{\partial v}{\partial \eta} + \frac{\partial^2 v}{\partial \xi^2} \right). \end{aligned}$$

The canonical form of the Tricomi equation is given by

$$\frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \xi^2} = -\frac{1}{3\eta} \frac{\partial v}{\partial \eta}.$$

3.4 General solutions

In general, it is not so simple to determine the general solution of a given equation. Sometimes further simplification of the canonical form of an equation may yield the general solution. If the canonical form of the equation is simple, then the general solution can be immediately ascertained.

Example 19. Find the general solution of the following equations

$$yu_{xx} + 3yu_{xy} + 3u_x = 0, \quad y \neq 0$$

Solution. First we have $A = y$, $B = 3y$ and $C = 0$. Hence $\Delta(x, y) = B^2 - 4AC = 9y^2 > 0$ and the equation is hyperbolic for all points (x, y) with $y \neq 0$. Consequently, the characteristic equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{3y \pm 3y}{2y} = 3, \text{ or } 0.$$

Integrating gives

$$y = c_1 \text{ and } y = 3x + c_2$$

where c_1 and c_2 are arbitrary constants. In terms of these variables, the canonical form of the equation is

$$\xi v_{\xi\eta} + v_\eta = 0.$$

Writing $w = v_\eta$ and using the integrating factor gives

$$w = v_\eta = \frac{1}{\xi} C(\eta)$$

where $C(\eta)$ is an arbitrary function. Integrating again with respect to η gives

$$v(\xi, \eta) = \frac{1}{\xi} \int C(\eta) d\eta + g(\xi) = \frac{1}{\xi} f(\eta) + g(\xi)$$

where f and g are arbitrary functions. Finally, in terms of the original variables, the general solution is

$$u(x, y) = \frac{1}{y} f(y - 3x) + g(y).$$

Example 20. If it is possible, find the general solution of the following equation

$$u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0.$$

Solution. The coefficients of equation are $A = 1$, $B = 2$, $C = 5$, $E = 1$, $F = G = 0$ and hence $B^2 - 4AC = -16 < 0$, equation is elliptic. The characteristic equations are

$$\frac{dy}{dx} = 1 \pm 2i$$

The curve characteristic is $\varphi(x, y) = y - (1 \pm 2i)x = C$, where C is an arbitrary constant. Then

we set $\xi = y - x$ and $\eta = -2x$ and by using the chain Rule, the canonical form is given by

$$v_{\xi\xi} + v_{\eta\eta} = \frac{1}{4}(v_{\xi} - 2v_{\eta}).$$

It is not easy to find a general solution for this equation.

3.5 Exercises

Exercise 3.1. Determine the region in which the given equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form.

- (1) $xu_{xx} + u_{yy} = x^2$;
- (2) $u_{xx} + y^2u_{yy} = y$;
- (3) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = e^x$;
- (4) $u_{xx}\sqrt{y}u_{xy} + xu_{yy} = \cos(x^2 - 2y)$, $y \geq 0$;
- (5) $\sin^2 xu_{xx} + \sin 2xu_{xy} + \cos^2 xu_{yy} = x$.

Exercise 3.2. Obtain the general solution of the following equations:

- (1) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + xyu_x + y^2u_y = 0$;
- (2) $ru_{tt} - c^2ru_{rr} - 2c^2u_r = 0$; $c = \text{constant}$
- (3) $u_{xx} + u_{yy} = 0$;
- (4) $u_{xx} + 2u_{xy} + u_{yy} = 0$;
- (5) $u_{xx} + 4u_{xy} + 4xu_{yy} = 0$.

Exercise 3.3. Find the characteristics and characteristic coordinates, and reduce the following equations to canonical form:

- (1) $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$;
- (2) $x^2u_{xx} - y^2u_{xy} - u_x = 1 + 2y^2$;
- (3) $x^2y^2u_{xx} + 2xy + u_{yy} = 0$;
- (4) $u_{yy} - 9u_x + 7u_y = \cos y$;
- (5) $u_{xy} + u_x + u_y = 3x$.

Exercise 3.4. Transform the following equations to the form $v_{\xi\eta} = cv$, $c = \text{constant}$

$$(1) \quad u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0;$$

$$(2) \quad 3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$$

by introducing the new variables $v = u e^{-(a\xi + b\eta)}$, where a and b are undetermined coefficients.

Exercise 3.5. Obtain the solution of the Cauchy problem

$$u_{xx} + u_{yy} = 0;$$

$$u(x, 0) = f(x) \quad u_y(x, 0) = g(x).$$

Exercise 3.6. Transform the equation

$$u_{xy} + yu_{yy} + \sin(x + y) = 0$$

into the canonical form. Use the canonical form to find the general solution.

Chapter 4

Method of separation of variables

4.1 Introduction

The method of separation of variables combined with the principle of super-position is widely used to solve initial boundary-value problems involving linear partial differential equations. Usually, the dependent variable $u(x, y)$ is expressed in the separable form $u(x, y) = X(x)Y(y)$, where X and Y are functions of x and y respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y . A similar treatment can be applied to equations in three or more independent variables. However, the question of separability of a partial differential equation into two or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding solutions of a large class of initial boundary-value problems. This method of solution is also known as *the Fourier method* (or *the method of eigenfunction expansion*). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems.

4.2 Separation of variables

Fourier's method for solving the heat equation provides a convenient method that can be applied to many other important linear problems. The method also enables us to deduce several properties of the solutions, such as asymptotic behavior, smoothness, and well-posedness. Historically, Fourier's idea was a breakthrough which paved the way for new developments in science and technology. For example, Fourier analysis found many applications in pure mathematics (number theory, approximation theory, etc.). Several fundamental theories in physics (quantum mechanics

in particular) are heavily based on Fourier's idea, and the entire theory of signal processing is based on Fourier's method and its generalizations.

Nevertheless, Fourier's method cannot always be applied for solving linear differential problems. The method is applicable only for problems with an appropriate symmetry. Moreover, the equation and the domain should share the same symmetry, and in most cases the domain should be bounded. Another drawback follows from the representation of the solution as an infinite series. In many cases it is not easy to prove that the formal solution given by this method is indeed a proper solution. Finally, even in the case when one can prove that the series converges to a classical solution, it might happen that the rate of convergence is very slow. Therefore, such a representation of the solution may not always be practical.

Fourier's method for solving linear PDEs is based on the technique of separation of variables. Let us outline the main steps of this technique. **First step**, we search for solutions of the homogeneous PDE that are **called product solutions** (or **separated solutions**). These solutions have the special form

$$u(x, t) = X(x)T(t),$$

and in general they should satisfy certain additional conditions. In many cases, these additional conditions are just homogeneous boundary conditions. It turns out that X and T should be solutions of linear ODEs that are easily derived from the given PDE. In **the second step**, we use a generalization of the superposition principle to generate out of the separated solutions a more general solution of the PDE, in the form of an infinite series of product solutions. In **the last step** we compute the coefficients of this series.

The following examples illustrate the general nature of this method of solution.

4.3 Examples

4.3.1 Heat equation: homogeneous boundary conditions

Consider the following heat conduction problem in a finite interval:

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < L, \quad t > 0, \quad (4.1)$$

with Dirichlet boundary conditions

$$u(0, t) = u(L, t) = 0, \quad t \geq 0, \quad (4.2)$$

and the initial condition

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (4.3)$$

where f is a given function and k is a positive constant. In order to make (4.2) consistent with (4.3), we assume the compatibility condition $f(0) = f(L) = 0$.

Step 1: Our goal is to find the solution of (4.1)–(4.3) using the method of separation of variables or Fourier method. A separable solution is a solution of the form

$$u(x, t) = X(x)T(t) \quad (4.4)$$

where X and T are functions of the variables x and t , respectively. At this step we do not take into account the initial condition (4.3). Obviously, we are not interested in the zero solution $u(x, t) = 0$. Therefore, we seek functions X and T that do not vanish identically.

Differentiate the separated solution (4.4) once with respect to t and twice with respect to x and substitute these derivatives into the PDE. We then obtain

$$X(x)T'(t) = kX''(x)T(t), \quad 0 < x < L, \quad t > 0.$$

Now, we carry out a simple but decisive step *the separation of variables step*. We move to one side of the PDE all the functions that depend only on x and to the other side the functions that depend only on t . We thus write

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}, \quad 0 < x < L, \quad t > 0. \quad (4.5)$$

Since x and t are independent variables, differentiating (4.5) with respect to t implies that there exists a constant denoted by λ (which is called the separation constant) such that

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda, \quad 0 < x < L, \quad t > 0. \quad (4.6)$$

Equation (4.6) leads to the following system of ODEs:

$$\frac{d^2X(x)}{dx^2} = -\lambda X(x), \quad 0 < x < L, \quad (4.7)$$

$$\frac{dT(t)}{dt} = k\lambda T(t), \quad t > 0, \quad (4.8)$$

which are coupled only by the separation constant λ . The function u satisfies the boundary

conditions (4.2) if and only if

$$\begin{cases} u(0, t) = X(0)T(t) = 0 \\ u(L, t) = X(L)T(t) = 0 \end{cases}$$

Since u is not the trivial solution $u = 0$, it follows that $X(0) = X(L) = 0$.

Therefore, the function X should be a solution of the boundary value problem

$$\begin{cases} \frac{d^2 X(x)}{dx^2} + \lambda X(x) = 0, & 0 < x < L, \\ X(0) = X(L) = 0. \end{cases} \quad (4.9)$$

Consider the system (4.9). A nontrivial solution of this system is called an eigenfunction of the problem with an eigenvalue λ . The problem (4.9) is called an **eigenvalue problem**. The boundary condition $X(0) = X(L) = 0$ is called (as in the PDE case) **the Dirichlet boundary condition**.

Note that the problem (4.9) is not an initial boundary problem for an ODE (for which it is known that there exists a unique solution). Rather, it is a boundary value problem for an ODE. It is not clear a priori that there exists a solution for any value of λ . On the other hand, if we can write the general solution of the ODE for every λ , then we need only to check for which λ there exists a solution that also satisfies the boundary conditions.

Fortunately, the first equation of (4.9) is quite elementary. It is a second-order linear ODE with constant coefficients, and its general solution (which depends on λ) has the following form:

- If $\lambda < 0$, then $X(x) = C_1 \cosh(\sqrt{-\lambda} x) + C_2 \sinh(\sqrt{-\lambda} x) = C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x}$.
- If $\lambda = 0$, then $X(x) = C_1 + C_2 x$.
- If $\lambda > 0$, then $X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$,

where C_1 and C_2 are arbitrary constants.

We begin by solving the system (4.9).

A nontrivial solution of (4.9) is called an eigenfunction with eigenvalue λ . We distinguish three cases:

Negative eigenvalue $\lambda = -\mu^2 < 0$. The general solution can be written

$$X(x) = \alpha e^{-\mu x} + \beta e^{\mu x}$$

where α, β are arbitrary real constants. From the boundary conditions, we have

$$\begin{cases} \alpha + \beta = 0, \\ \alpha e^{-\mu L} + \beta e^{\mu L} = 0. \end{cases}$$

From the first equation, we have $\alpha = -\beta$. Substituting into the second gives $\alpha e^{-\mu L} = \alpha e^{\mu L}$, and if $\alpha \neq 0$ then $e^{2\mu L} = 1$. This is not possible since μ and L are both nonzero. Hence, $\alpha = \beta = 0$. So, in this case, $X \equiv 0$ and $u(x, t) = 0$ for all $0 \leq x \leq L$ and $t \geq 0$. Thus, the system (4.9) does not admit a negative eigenvalue.

Zero eigenvalue $\lambda = 0$, then, the general solution for is

$$X(x) = \alpha + \beta x,$$

where α, β are arbitrary real constants. The boundary conditions imply:

$$\begin{cases} \alpha + \beta \cdot 0 = 0, \\ \alpha + \beta L = 0. \end{cases}$$

Since $L \neq 0$, it follows clearly that $\alpha = \beta = 0$. Thus, again $X(x) \equiv 0$ and $u(x, t) = 0$ for all $0 \leq x \leq L$ and $t \geq 0$. We must also exclude the case $\lambda = 0$.

Positive eigenvalue $\lambda = \mu^2 > 0$. The general solution is

$$X(x) = \alpha \cos(\mu x) + \beta \sin(\mu x)$$

where α, β are arbitrary real constants. Substituting this solution into the boundary condition, we obtain

$$\begin{cases} \alpha = 0, \\ \beta \sin(\mu L) = 0 \end{cases}$$

If $\beta = 0$, the solution is trivial. For nontrivial solutions, $\beta \neq 0$, hence, $\sin(\mu L) = 0$. Consequently,

$$\mu L = n\pi, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{Z}$$

It follows that

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

are the eigenvalues, and the corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{Z}.$$

Since $\sin(-x) = -\sin(x)$ for all $x \in \mathbb{R}$, the solutions of problems (4.9) are, therefore

$$X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}^*.$$

Let us deal now with the ODE (4.8). For $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, the general solution has the form

$$T(t) = \gamma_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{N}^*.$$

where γ_n is an arbitrary constant.

Step 2: We have thus obtained the following sequence of separated solutions

$$u_n(x, t) = \delta_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{N}^*.$$

where $\delta_n = \beta_n \gamma_n$.

Since equation (4.1) is linear and homogeneous, by the superposition principle, the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} \delta_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

of separated solutions is also a solution of the heat equation that satisfies the Dirichlet boundary conditions.

Consider now the initial condition, we have

$$u(x, 0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} \delta_n \sin\left(\frac{n\pi x}{L}\right)$$

Step 3: In other words, it is possible to find constants δ_n such that:

$$f(x) = \sum_{n=1}^{\infty} \delta_n \sin\left(\frac{n\pi x}{L}\right)$$

Such a series is called a (*generalized*) Fourier series (or *expansion*) of the function f with respect to the eigenfunctions of the problem, and δ_n , for $n = 1, 2, \dots$ are called the (*generalized*) Fourier coefficients of the series.

Observe that:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x) dx = \sum_{n=1}^{\infty} \delta_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

Therefore, the Fourier coefficients are given by:

$$\delta_n = \frac{\int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x) dx}{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx} = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) dx.$$

Finally, the explicit formula of the solution of the problem (4.1)–(4.3), is given by:

$$u(x, t) = \sum_{n=1}^{\infty} \delta_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where

$$\delta_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) dx.$$

Example 21. Consider the problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \pi, \quad t > 0, \quad (4.10)$$

with Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0, \quad (4.11)$$

and the initial condition

$$u(x, 0) = f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2}; t > 0 \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi. \end{cases} \quad (4.12)$$

Solution. The formal solution is

$$u(x, t) = \sum_{n=1}^{\infty} \delta_n \sin(nx) e^{-n^2 t}$$

where

$$\delta_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x) dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin(nx) dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin(nx) dx = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}.$$

But

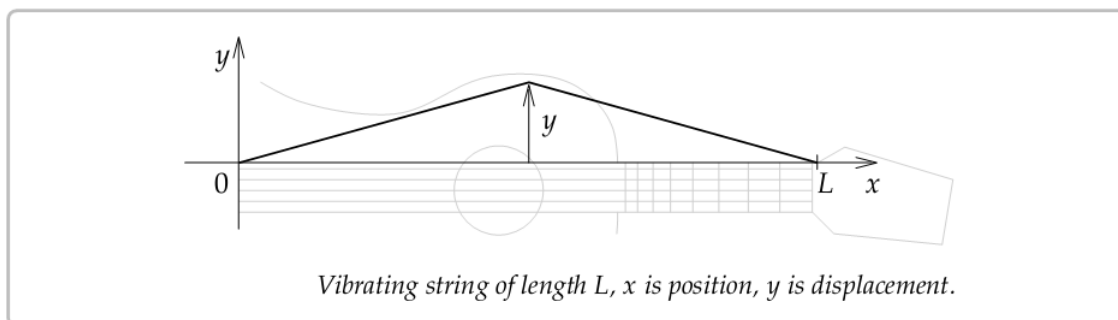
$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n = 2m, \\ (-1)^{m+1} & n = 2m - 1, \end{cases}$$

when $m = 1, 2, \dots$. Therefore the formal solution is

$$u(x, t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin[(2m-1)x] e^{-(2m-1)^2 t}.$$

4.3.2 One-dimensional wave equation

Imagine a tensioned guitar string of length L that can vibrate. We will only consider vibrations in one direction. Let x denote the position along the string, let t denote time, and let $y(x, t)$ denote the displacement of the string from the rest position. See The equation that governs this setup is



the so-called *one-dimensional wave equation*:

$$y_{tt} = a^2 y_{xx},$$

for some constant $a > 0$.

The intuition is similar to the heat equation, replacing velocity with acceleration: the acceleration at a specific point is proportional to the second derivative of the shape of the string. In other words, when the string is concave down then y_{xx} is negative and the string wants to accelerate downwards, so y_{tt} should be negative. And vice versa. The wave equation is an example of a hyperbolic PDE.

We will again solve for y in the region $0 < x < L$ and $t > 0$. Assume that the ends of the string are fixed in place as on the guitar:

$$y(0, t) = 0 \quad \text{and} \quad y(L, t) = 0 \quad \text{for } t > 0.$$

We have two conditions along the x -axis as there are two derivatives in the x direction. There are also two derivatives along the t direction and hence we need two further conditions here. We need to know the initial position and the initial velocity of the string. That is, for some known functions $f(x)$ and $g(x)$, we impose

$$y(x, 0) = f(x) \quad \text{and} \quad y_t(x, 0) = g(x) \quad \text{for } 0 < x < L.$$

The equation is linear and homogeneous, so superposition works just as it did for the heat equation. Superposition also preserves the homogeneous side conditions $y(0, t) = 0$ and $y(L, t) = 0$. Again we will use separation of variables to find enough building-block solutions to get the particular solution also solving the nonhomogeneous initial conditions. There is one change however. We will solve two separate problems and add their solutions.

The two problems we will solve are

$$\begin{aligned} w_{tt} &= a^2 w_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ w(0, t) &= w(L, t) = 0 && \text{for } t > 0, \\ w(x, 0) &= 0 && \text{for } 0 < x < L, \\ w_t(x, 0) &= g(x) && \text{for } 0 < x < L, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} z_{tt} &= a^2 z_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ z(0, t) &= z(L, t) = 0 && \text{for } t > 0, \\ z(x, 0) &= f(x) && \text{for } 0 < x < L, \\ z_t(x, 0) &= 0 && \text{for } 0 < x < L. \end{aligned} \tag{4.14}$$

The principle of superposition implies that $y = w + z$ solves the wave equation and the homogeneous side conditions. Furthermore, $y(x, 0) = w(x, 0) + z(x, 0) = f(x)$ and $y_t(x, 0) = w_t(x, 0) + z_t(x, 0) = g(x)$. Hence, y is a solution to

$$\begin{aligned} y_{tt} &= a^2 y_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ y(0, t) &= y(L, t) = 0 && \text{for } t > 0, \\ y(x, 0) &= f(x) && \text{for } 0 < x < L, \\ y_t(x, 0) &= g(x) && \text{for } 0 < x < L. \end{aligned} \tag{4.15}$$

The reason for all this complexity is that superposition only works for homogeneous conditions such as $y(0, t) = y(L, t) = 0$, $y(x, 0) = 0$, or $y_t(x, 0) = 0$. Therefore, we can use separation of

variables to find many building-block solutions solving all the homogeneous conditions. We can then use them to construct a solution satisfying the remaining nonhomogeneous condition.

Let us start with (4.13). We try a solution of the form $w(x, t) = X(x)T(t)$ again. We plug into the wave equation to obtain

$$X(x)T''(t) = a^2X''(x)T(t).$$

Rewriting, we get

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)}.$$

Again, left-hand side depends only on t and the right-hand side depends only on x . So both sides equal a constant, which we denote by $-\lambda$:

$$\frac{T''(t)}{a^2T(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

We solve to get two ordinary differential equations

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T''(t) + \lambda a^2 T(t) &= 0. \end{aligned}$$

The condition $0 = w(0, t) = X(0)T(t)$ implies $X(0) = 0$ and $w(L, t) = 0$ implies that $X(L) = 0$. Therefore, the only nontrivial solutions for the first equation are when $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$ and they are

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

The general solution for T for this particular λ_n is

$$T_n(t) = A \cos\left(\frac{n\pi a}{L}t\right) + B \sin\left(\frac{n\pi a}{L}t\right).$$

We also have the condition that $w(x, 0) = 0$ or $X(x)T(0) = 0$. This implies that $T(0) = 0$, which in turn forces $A = 0$. It is convenient to pick $B = \frac{L}{n\pi a}$ (you will see why in a moment) and hence

$$T_n(t) = \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right).$$

Our building-block solutions are

$$w_n(x, t) = \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right).$$

We differentiate in t :

$$\frac{\partial w_n}{\partial t}(x, t) = \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

Hence,

$$\frac{\partial w_n}{\partial t}(x, 0) = \sin\left(\frac{n\pi}{L}x\right).$$

We expand $g(x)$ in terms of these sines as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

Using superposition we write the solution to (4.13) as a series

$$w(x, t) = \sum_{n=1}^{\infty} b_n w_n(x, t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right).$$

We solve (4.14) similarly. We again try $z(x, y) = X(x)T(t)$. The procedure works exactly the same at first. We obtain

$$X''(x) + \lambda X(x) = 0,$$

$$T''(t) + \lambda a^2 T(t) = 0,$$

and the conditions $X(0) = 0$, $X(L) = 0$. Again, $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

This time, the condition on T is $T'(0) = 0$. Thus we get that $B = 0$, and we take

$$T_n(t) = \cos\left(\frac{n\pi a}{L}t\right).$$

Our building-block solution is

$$z_n(x, t) = \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

As $z_n(x, 0) = \sin\left(\frac{n\pi}{L}x\right)$, we expand $f(x)$ in terms of these sines as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right).$$

We write down the solution to (4.14) as a series

$$z(x, t) = \sum_{n=1}^{\infty} c_n z_n(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right).$$

Putting these two solutions together, let us state the result as a theorem.

Theorem 9. *Take the problem*

$$\begin{aligned} y_{tt} &= a^2 y_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ y(0, t) &= y(L, t) = 0 && \text{for } t > 0, \\ y(x, 0) &= f(x) && \text{for } 0 < x < L, \\ y_t(x, 0) &= g(x) && \text{for } 0 < x < L, \end{aligned} \tag{4.16}$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

Then the solution $y(x, t)$ can be written as a sum of the solutions of (4.13) and (4.14):

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right) + c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + c_n \cos\left(\frac{n\pi a}{L}t\right) \right]. \end{aligned}$$

4.4 Exercises

Exercise 4.1. *Using the method of separation of variables, find the solution of the following problem*

$$\begin{aligned} u_t &= k u_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ u_x(0, t) &= u_x(L, t) = 0 && \text{for } t > 0, \\ u(x, 0) &= f(x) && \text{for } 0 < x < L. \end{aligned}$$

Exercise 4.2. *Using the method of separation of variables, find the solution of the following problem*

$$\begin{aligned} u_t &= k u_{xx} && \text{for } 0 < x < L \text{ and } t > 0, \\ u(0, t) &= u(L, t), \quad u_x(0, t) = u_x(L, t) && \text{for } t > 0, \\ u(x, 0) &= f(x) && \text{for } 0 < x < L. \end{aligned}$$

Exercise 4.3. *Imagine that a stringed musical instrument falls on the floor. Suppose that the length of the string is 1 and $a = 1$. When the musical instrument hits the ground the string was*

in rest position and hence $y(x, 0) = 0$. However, the string was moving at some velocity at impact ($t = 0$), say $y_t(x, 0) = -1$. Find the solution $y(x, t)$ for the shape of the string at time t .

Exercise 4.4. Suppose that you have a vibrating string and that there is air resistance proportional to the velocity. That is, you have

$$y_{tt} = a^2 y_{xx} - ky_t \quad \text{for } 0 < x < 1 \text{ and } t > 0,$$

$$y(0, t) = y(1, t) = 0 \quad \text{for } t > 0,$$

$$y(x, 0) = f(x) \quad \text{for } 0 < x < 1,$$

$$y_t(x, 0) = 0 \quad \text{for } 0 < x < 1.$$

Suppose that $0 < k < 2\pi a$. Derive a series solution to the problem. Any coefficients in the series should be expressed as integrals of $f(x)$.

Chapter 5

Laplace's Equation

5.1 Introduction

Consider an insulated wire, a plate, or a 3-dimensional object. We apply certain fixed temperatures on the ends of the wire, the edges of the plate, or on all sides of the 3-dimensional object. We wish to find out what is the *steady-state temperature* distribution. That is, we wish to know what will be the temperature after long enough period of time.

We are really seeking a solution to the heat equation that is not dependent on time. We first solve the problem in one space variable. We are looking for a function u that satisfies

$$u_t = ku_{xx},$$

but such that $u_t = 0$ for all x and t . Hence, we are looking for a function of x alone that satisfies $u_{xx} = 0$. It is easy to solve this equation by integration, and we see that $u = Ax + B$ for some constants A and B .

Consider an insulated wire where we apply constant temperature T_1 at one end (say where $x = 0$) and T_2 on the other end (at $x = L$ where L is the length of the wire). Our steady-state solution is

$$u(x) = \frac{T_2 - T_1}{L}x + T_1.$$

It is simply a straight line from one end to the other. This solution agrees with the common-sense intuition on how heat should be distributed in the wire. So in one dimension, the steady-state solutions are just straight lines.

Things are more complicated in two or more space dimensions. We restrict ourselves to two

space dimensions for simplicity. The heat equation in two space variables is

$$u_t = k(u_{xx} + u_{yy}), \quad (5.1)$$

or more commonly written as $u_t = k\Delta u$ or $u_t = k\nabla^2 u$. The Δ and ∇^2 symbols both mean $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We will use Δ here. The reason for using such a notation is that you can define Δ to be the right thing for any number of space dimensions and then the heat equation is always $u_t = k\Delta u$. The operator Δ is called the *Laplacian*.

OK, now that we have notation out of the way, let us see what does an equation for the steady-state solution look like. We are looking for a solution to (5.1) that does not depend on t , that is, $u_t = 0$.

Hence, we are looking for a function $u(x, y)$ such that

$$\Delta u = u_{xx} + u_{yy} = 0.$$

This equation is called the *Laplace equation*¹ and is an example of an elliptic equation. Solutions to the Laplace equation are called *harmonic functions* and have many nice properties and applications far beyond the steady-state heat problem.

Definition 16. Let Ω be a bounded regular open set in \mathbb{R}^n with $n \geq 1$ and $\partial\Omega$ its boundary. If the unknown $u(\mathbf{x})$, ($\mathbf{x} = (x_1, \dots, x_n) \in \Omega$), satisfies the following equation in Ω :

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega, \quad (5.2)$$

where Δ denotes the partial differential operator, that is:

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Then, equation (5.2) is called *Laplace's equation* in Ω .

5.2 Harmonic Functions

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $\partial\Omega$ its boundary. In the following, we denote by $\mathbf{x} = (x_1, \dots, x_n)$.

¹Named after the French mathematician [Pierre-Simon, marquis de Laplace](#) (1749–1827).

Definition 17. Let $u \in C^2(\Omega)$ be a real-valued function. We say that u is harmonic if:

$$\Delta u(\mathbf{x}) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(\mathbf{x}) = 0.$$

Example 22. Show that the function u defined by

$$u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$

is a harmonic function on the disk $D = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$.

5.3 Dirichlet problem in the circle and the Poisson kernel

A more natural setting for the Laplace equation $\Delta u = 0$ is a circle rather than a rectangle. On the other hand, what makes the problem somewhat more difficult is that we need polar coordinates.

Recall that the polar coordinates for the (x, y) -plane are (r, θ) :

$$x = r \cos \theta, \quad y = r \sin \theta,$$

where $r \geq 0$ and $-\pi < \theta \leq \pi$.

So the point (x, y) is distance r from the origin at an angle θ from the positive x -axis.

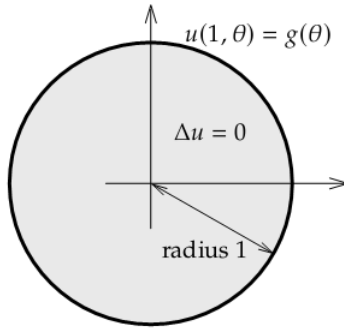
Now that we know our coordinates, let us give the problem we wish to solve. We have a circular region of radius 1, and we are interested in the Dirichlet problem for the Laplace equation for this region. Let $u(r, \theta)$ denote the temperature at the point (r, θ) in polar coordinates.

We have the problem:

$$\begin{aligned} \Delta u &= 0 && \text{for } r < 1, \\ u(1, \theta) &= g(\theta) && \text{for } -\pi < \theta \leq \pi. \end{aligned} \tag{5.3}$$

The first issue we face is that we do not know the Laplacian in polar coordinates. Normally, we would find u_{xx} and u_{yy} in terms of the derivatives in r and θ . We would need to solve for r and θ in terms of x and y . In this case, it is more convenient to work in reverse. We compute derivatives in r and θ in terms of derivatives in x and y and then we solve. The computations are easier this way. First,

$$\begin{aligned} x_r &= \cos \theta, & x_\theta &= -r \sin \theta, \\ y_r &= \sin \theta, & y_\theta &= r \cos \theta. \end{aligned}$$



By chain rule, we obtain

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = \cos(\theta)u_x + \sin(\theta)u_y, \\ u_{rr} &= \cos(\theta)(u_{xx}x_r + u_{xy}y_r) + \sin(\theta)(u_{yx}x_r + u_{yy}y_r) \\ &= \cos^2(\theta)u_{xx} + 2\cos(\theta)\sin(\theta)u_{xy} + \sin^2(\theta)u_{yy}. \end{aligned}$$

Similarly for the θ derivative. Note that we have to use the product rule for the second derivative.

$$\begin{aligned} u_\theta &= u_x x_\theta + u_y y_\theta = -r \sin(\theta)u_x + r \cos(\theta)u_y, \\ u_{\theta\theta} &= -r \cos(\theta)u_x - r \sin(\theta)(u_{xx}x_\theta + u_{xy}y_\theta) - r \sin(\theta)u_y + r \cos(\theta)(u_{yx}x_\theta + u_{yy}y_\theta) \\ &= -r \cos(\theta)u_x - r \sin(\theta)u_y + r^2 \sin^2(\theta)u_{xx} - r^2 2 \sin(\theta) \cos(\theta)u_{xy} + r^2 \cos^2(\theta)u_{yy}. \end{aligned}$$

Let us now try to find $u_{xx} + u_{yy}$. We start with $\frac{1}{r^2}u_{\theta\theta}$ to get rid of those pesky r^2 . If we add u_{rr} and use the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$, we get

$$\frac{1}{r^2}u_{\theta\theta} + u_{rr} = u_{xx} + u_{yy} - \frac{1}{r} \cos(\theta)u_x - \frac{1}{r} \sin(\theta)u_y.$$

We are not quite there yet, but all we are lacking is $\frac{1}{r}u_r$. We add it to obtain the *Laplacian in polar coordinates*:

$$\Delta u = u_{xx} + u_{yy} = \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr}.$$

Notice that the Laplacian in polar coordinates no longer has constant coefficients.

5.3.1 Series solution

Let us separate variables as usual. That is, we try $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$0 = \Delta u = \frac{1}{r^2}R\Theta'' + \frac{1}{r}R'\Theta + R''\Theta.$$

We put R on one side and Θ on the other and conclude that both sides must be constant.

$$\begin{aligned} \frac{1}{r^2}R\Theta'' &= -\left(\frac{1}{r}R' + R''\right)\Theta \\ \frac{\Theta''}{\Theta} &= -\frac{rR' + r^2R''}{R} = -\lambda \end{aligned}$$

We get two equations:

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0, \\ r^2R'' + rR' - \lambda R &= 0. \end{aligned}$$

We first focus on Θ . We know that $u(r, \theta)$ ought to be 2π -periodic in θ , that is, $u(r, \theta) = u(r, \theta + 2\pi)$.

Therefore, the solution to $\Theta'' + \lambda\Theta = 0$ must be 2π -periodic.

We conclude that $\lambda = n^2$ for a nonnegative integer $n = 0, 1, 2, 3, \dots$. The equation becomes $\Theta'' + n^2\Theta = 0$. When $n = 0$ the equation is just $\Theta'' = 0$, so we have the general solution $A\theta + B$. As Θ is periodic, $A = 0$. For convenience, we write this solution as

$$\Theta_0 = \frac{a_0}{2}$$

for some constant a_0 . For positive n , the solution to $\Theta'' + n^2\Theta = 0$ is

$$\Theta_n = a_n \cos(n\theta) + b_n \sin(n\theta),$$

for some constants a_n and b_n .

Next, we consider the equation for R ,

$$r^2R'' + rR' - n^2R = 0.$$

The idea is to try a solution r^s and if that does not give us two solutions, also try a solution of

the form $r^s \ln r$. We name the solution R_n as usual. When $n = 0$ we obtain

$$R_0 = Ar^0 + Br^0 \ln r = A + B \ln r,$$

and if $n > 0$, we get

$$R_n = Ar^n + Br^{-n}.$$

The function $u(r, \theta)$ must be finite (it cannot blow up) at the origin, that is, when $r = 0$. So $B = 0$ in both cases as otherwise r^{-n} or $\ln r$ does blow up as $r \rightarrow 0$. Set $A = 1$ in both cases; the constants in Θ_n will pick up the slack so nothing is lost. That is,

$$R_0 = 1 \quad \text{and} \quad R_n = r^n.$$

Our building block solutions are

$$u_0(r, \theta) = \frac{a_0}{2} \quad \text{and} \quad u_n(r, \theta) = a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

Putting everything together our solution is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

We look at the boundary condition in (5.3),

$$g(\theta) = u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Therefore, to solve (5.3) we expand $g(\theta)$, which is a 2π -periodic function, as a Fourier series, and then multiply the n^{th} term by r^n . To find the a_n and the b_n , we compute

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta.$$

5.3.2 Poisson kernel

There is another way to solve the Dirichlet problem—with the help of an integral kernel. That is, we will find a function $P(r, \theta, \alpha)$ called the *Poisson kernel*² such that

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta, \alpha) g(\alpha) d\alpha.$$

While the integral will generally not be solvable analytically, it can be evaluated numerically. In fact, unless the boundary data is given as a Fourier series already, it may be much easier to numerically evaluate this formula as there is only one integral to evaluate.

The formula also has theoretical applications. For instance, as $P(r, \theta, \alpha)$ will have infinitely many derivatives, then via differentiating under the integral, we find that the solution $u(r, \theta)$ has infinitely many derivatives, at least when inside the circle, $r < 1$. By *having infinitely many derivatives*, what you should think of is that $u(r, \theta)$ has *no corners* and all of its partial derivatives of all orders exist and also have *no corners*.

We will compute the formula for $P(r, \theta, \alpha)$ from the series solution, and this idea can be applied anytime you have a convenient series solution where the coefficients are obtained via integration. Hence you can apply this reasoning to obtain such integral kernels for other equations, such as the heat equation. The computation is long and *tedious*, but not overly difficult. Since the ideas are often applied in similar contexts, it is good to understand how this computation works.

What we do is start with the series solution and replace the coefficients with the integrals that compute them. Then we try to write everything as a single integral. We must use a different

²Named for the French mathematician [Siméon Denis Poisson](#) (1781–1840).

dummy variable for the integration and hence we use α instead of θ .

$$\begin{aligned}
 u(r, \theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\
 &= \underbrace{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\alpha) d\alpha \right)}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \cos(n\alpha) d\alpha \right)}_{a_n} r^n \cos(n\theta) + \\
 &\quad + \underbrace{\left(\frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \sin(n\alpha) d\alpha \right)}_{b_n} r^n \sin(n\theta) \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(g(\alpha) + 2 \sum_{n=1}^{\infty} g(\alpha) \cos(n\alpha) r^n \cos(n\theta) + g(\alpha) \sin(n\alpha) r^n \sin(n\theta) \right) d\alpha \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(1 + 2 \sum_{n=1}^{\infty} r^n (\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)) \right)}_{P(r, \theta, \alpha)} g(\alpha) d\alpha.
 \end{aligned}$$

OK, so we have what we wanted, the expression in the parentheses is the Poisson kernel, $P(r, \theta, \alpha)$. However, we can do a lot better. It is still given as a series, and we would really like to have a nice simple expression for it. We must work a little harder. The trick is to rewrite everything in terms of complex exponentials. Let us work just on the kernel.

$$\begin{aligned}
 P(r, \theta, \alpha) &= 1 + 2 \sum_{n=1}^{\infty} r^n (\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)) \\
 &= 1 + 2 \sum_{n=1}^{\infty} r^n \cos(n(\theta - \alpha)) \\
 &= 1 + \sum_{n=1}^{\infty} r^n (e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}) \\
 &= 1 + \sum_{n=1}^{\infty} (r e^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty} (r e^{-i(\theta-\alpha)})^n.
 \end{aligned}$$

In the expression above, we recognize the *geometric series*. Recall from calculus that if z is a complex number where $|z| < 1$, then

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Note that n starts at 1, and that is why we have the z in the numerator. It is the standard geometric series multiplied by z . We can use $z = r e^{i(\theta-\alpha)}$, as lo and behold $|r e^{i(\theta-\alpha)}| = r < 1$. We

continue with the computation.

$$\begin{aligned}
 P(r, \theta, \alpha) &= 1 + \sum_{n=1}^{\infty} (re^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty} (re^{-i(\theta-\alpha)})^n \\
 &= 1 + \frac{re^{i(\theta-\alpha)}}{1 - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{1 - re^{-i(\theta-\alpha)}} \\
 &= \frac{(1 - re^{i(\theta-\alpha)})(1 - re^{-i(\theta-\alpha)}) + (1 - re^{-i(\theta-\alpha)})re^{i(\theta-\alpha)} + (1 - re^{i(\theta-\alpha)})re^{-i(\theta-\alpha)}}{(1 - re^{i(\theta-\alpha)})(1 - re^{-i(\theta-\alpha)})} \\
 &= \frac{1 - r^2}{1 - re^{i(\theta-\alpha)} - re^{-i(\theta-\alpha)} + r^2} \\
 &= \frac{1 - r^2}{1 - 2r \cos(\theta - \alpha) + r^2}.
 \end{aligned}$$

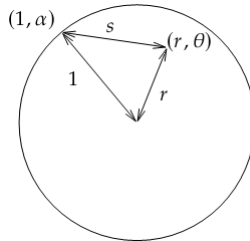
That is a formula we can live with. The solution to the Dirichlet problem using the Poisson kernel is

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \alpha) + r^2} g(\alpha) d\alpha.$$

Sometimes the formula for the Poisson kernel is given together with the constant $\frac{1}{2\pi}$, in which case we should, of course, not leave it in front of the integral. Sometimes the limits of the integral are given as 0 to 2π ; everything inside is 2π -periodic in α , so this does not change the integral.

Let us not leave the Poisson kernel without explaining its geometric meaning. Let s be the distance from (r, θ) to $(1, \alpha)$. This distance s in polar coordinates is given precisely by the square root of $1 - 2r \cos(\theta - \alpha) + r^2$. That is, the Poisson kernel is really the formula

$$\frac{1 - r^2}{s^2}.$$



One final note we make about the formula is that it is really a weighted average of the boundary

values. First, we look at what happens at the origin, that is, when $r = 0$:

$$\begin{aligned} u(0, 0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - 0^2}{1 - 2(0) \cos(0 - \alpha) + 0^2} g(\alpha) d\alpha \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\alpha) d\alpha. \end{aligned}$$

So $u(0, 0)$ is precisely the average value of $g(\theta)$ and therefore the average value of u on the boundary. This is a general feature of harmonic functions, the value at some point p is equal to the average of the values on a circle centered at p .

What the formula says at other points inside the circle is that the value of the solution is a weighted average of the boundary data $g(\theta)$. The kernel is bigger when $(1, \alpha)$ is closer to (r, θ) . Therefore, when computing $u(r, \theta)$, we give more weight to the values $g(\alpha)$ when $(1, \alpha)$ is closer to (r, θ) and less weight to the values $g(\alpha)$ when $(1, \alpha)$ far from (r, θ) .

5.4 Exercises

Exercise 5.1. Using series solve $\Delta u = 0$, $u(1, \theta) = |\theta|$, for $-\pi < \theta \leq \pi$.

Exercise 5.2. Using series solve $\Delta u = 0$, $u(1, \theta) = g(\theta)$ for the following data. Hint: trig identities.

(i) $g(\theta) = \frac{1}{2} + 3 \sin(\theta) + \cos(3\theta)$

(ii) $g(\theta) = 3 \cos(3\theta) + 3 \sin(3\theta) + \sin(9\theta)$

(iii) $g(\theta) = 2 \cos(\theta + 1)$

(iv) $g(\theta) = \sin^2(\theta)$

Exercise 5.3. Using the Poisson kernel, give the solution to $\Delta u = 0$, where $u(1, \theta)$ is zero for θ outside the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ and $u(1, \theta)$ is 1 for θ on the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

Exercise 5.4. (i) Draw a graph for the Poisson kernel as a function of α when $r = \frac{1}{2}$ and $\theta = 0$.

(ii) Describe what happens to the graph when you make r bigger (as it approaches 1).

(iii) Knowing that the solution $u(r, \theta)$ is the weighted average of $g(\theta)$ with Poisson kernel as the weight, explain what your answer to part b) means.

Exercise 5.5. Let $g(\theta)$ be the function $xy = \cos\theta \sin\theta$ on the boundary. Use the series solution to find a solution to the Dirichlet problem $\Delta u = 0$, $u(1, \theta) = g(\theta)$. Now convert the solution to Cartesian coordinates x and y . Is this solution surprising? Hint: use your trig identities.

Exercise 5.6. Carry out the computation we needed in the separation of variables and solve $r^2 R'' + rR' - n^2 R = 0$, for $n = 0, 1, 2, 3, \dots$

Exercise 5.7. Derive the series solution to the Dirichlet problem if the region is a circle of radius ρ rather than 1. That is, solve $\Delta u = 0$, $u(\rho, \theta) = g(\theta)$.

Exercise 5.8. (i) Find the solution for $\Delta u = 0$, $u(1, \theta) = x^2 y^3 + 5x^2$. Write the answer in Cartesian coordinates.

(ii) Now solve $\Delta u = 0$, $u(1, \theta) = x^k y^\ell$. Write the solution in Cartesian coordinates.

(iii) Suppose you have a polynomial $P(x, y) = \sum_{j=0}^m \sum_{k=0}^n c_{j,k} x^j y^k$, solve $\Delta u = 0$, $u(1, \theta) = P(x, y)$ (that is, write down the formula for the answer). Write the answer in Cartesian coordinates.

Notice the answer is again a polynomial in x and y .

Exercise 5.9. Using series solve $\Delta u = 0$, $u(1, \theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\theta)$.

Exercise 5.10. Using the series solution find the solution to $\Delta u = 0$, $u(1, \theta) = 1 - \cos(\theta)$. Express the solution in Cartesian coordinates (that is, using x and y).

Exercise 5.11. (i) Try and guess a solution to $\Delta u = -1$, $u(1, \theta) = 0$. Hint: try a solution that only depends on r . Also first, don't worry about the boundary condition.

(ii) Now solve $\Delta u = -1$, $u(1, \theta) = \sin(2\theta)$ using superposition.

Exercise 5.12. Derive the Poisson kernel solution if the region is a circle of radius ρ rather than 1. That is, solve $\Delta u = 0$, $u(\rho, \theta) = g(\theta)$.

Chapter 6

Wave equation

In this chapter we study the one-dimensional wave equation on the real line. Moreover, we shall derive simple explicit formulas for the solutions. We also discuss some important properties of the solutions of the wave equation which are typical for more general hyperbolic problems as well. The wave equation is certainly one of the most important classical equations of mathematical physics.

6.1 The wave equation in one dimension $n = 1$

The inhomogeneous wave equation in one (spatial) dimension has the form

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = F(x, t)$$

where $c \in \mathbb{R}$ is called **the wave speed**. This is the equation of motion of a vibrating string, mathematically referred to as the one-dimensional d'Alembert equation.

Remark 7. *The homogeneous wave equation in n dimension is written as:*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - c^2 \Delta u(\mathbf{x}, t) = 0, & \forall \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}^+, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n, \\ \frac{\partial u}{\partial t}(\mathbf{x}, 0) = v_0(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

where c is the wave speed, and u_0 and v_0 are two given functions in $C^2(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$, respectively.

Definition 18. *The operator*

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$$

is called the wave operator, or the d'Alembertian in dimension n .

6.2 d'Alembert's formula for the wave equation

6.2.1 Homogeneous wave equation

The Cauchy problem for the one-dimensional homogeneous wave equation is given by

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}^+, \quad (6.1)$$

with the following initial conditions:

$$u(x, 0) = f(x), \quad \text{and} \quad \partial_t u(x, 0) = g(x), \quad x \in \mathbb{R}. \quad (6.2)$$

This problem is one special case where the general solution of the differential equation actually leads to a solution of the problem.

Theorem 10. *The function u defined on $\mathbb{R} \times \mathbb{R}$ by:*

$$u(x, t) = \frac{1}{2} \left[f(x - ct) + f(x + ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \quad (6.3)$$

is a solution to the problem (6.1)–(6.2).

Remark 8. *The formula (6.3) is called d'Alembert's formula for the homogeneous one-dimensional wave equation.*

Proof. We introduce the following change of variables:

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \quad (6.4)$$

Therefore:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \\ \frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right]. \end{cases} \quad (6.5)$$

Hence, using (6.5), the equation (6.1) is equivalent to:

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

and since $c \neq 0$, we have

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

The general solution of equation (6.6) is of the form

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

therefore:

$$u(x, t) = F(x + ct) + G(x - ct) \quad (6.6)$$

where F and G are arbitrary twice-differentiable functions of a single variable.

Now, we determine F and G such that the solution (6.6) satisfies the initial conditions (6.2).

Then, for $t = 0$ in (6.6), we obtain:

$$\begin{cases} F(x) + G(x) = f(x), \\ cF'(x) - cG'(x) = g(x), \end{cases} \quad \forall x \in \mathbb{R}$$

which implies

$$(S) \begin{cases} F'(x) + G'(x) = f'(x), \\ cF'(x) - cG'(x) = g(x), \end{cases} \quad \forall x \in \mathbb{R}.$$

Solving the linear system (S) for $F'(x)$ and $G'(x)$ yields:

$$\begin{cases} F'(x) = \frac{1}{2} [f'(x) + \frac{1}{c}g(x)], \\ G'(x) = \frac{1}{2} [f'(x) - \frac{1}{c}g(x)]. \end{cases} \quad (6.7)$$

Integrating both equations of (6.7) from 0 to x , we obtain:

$$\begin{cases} F(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_0^x g(s) ds + c^*, \\ G(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_0^x g(s) ds - c^*, \end{cases}$$

where c^* is an integration constant.

Now replacing x in the expressions for F and G by $x + ct$ and $x - ct$ respectively, and using

(6.6), we obtain:

$$\begin{aligned} u(x, t) &= F(x + ct) + G(x - ct) \\ &= \frac{1}{2} \left[f(x - ct) + f(x + ct) \right] + \frac{1}{2c} \left[\int_0^{x+ct} g(s) ds - \int_0^{x-ct} g(s) ds \right] \\ &= \frac{1}{2} \left[f(x - ct) + f(x + ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds. \end{aligned}$$

This completes the proof of Theorem 10. □

Example 23. Using d'Alembert's formula, determine the solution to equation (6.1) with the following initial conditions:

$$u(x, 0) = \sin x \quad \text{and} \quad u_t(x, 0) = 0.$$

Solution: According to formula (6.3), we have

$$u(x, t) = \frac{\sin(x - ct) + \sin(x + ct)}{2} = \sin x \cos(ct),$$

which is a solution of equation (6.1).

Example 24. Using d'Alembert's formula, solve equation (6.1) with the following initial conditions:

$$u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = \sin(2x).$$

Solution: According to formula (6.3), we have

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(2s) ds = \frac{\sin(2x) \sin(2ct)}{2c},$$

which is a solution of equation (6.1).

6.2.2 Non homogeneous wave equation

We consider the non homogeneous wave equation in one dimension defined over the entire real line:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = F(x, t), \quad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}^+, \quad (6.8)$$

with the following initial conditions:

$$u(x, 0) = \psi(x), \quad \text{and} \quad \partial_t u(x, 0) = \phi(x), \quad -\infty < x < +\infty, \quad (6.9)$$

where F is an arbitrary function.

Theorem 11. *Let $F \in C^0(\mathbb{R} \times \mathbb{R}_+)$. Consider the non homogeneous Cauchy problem (6.8)–(6.9) with $\psi \in C^2(\mathbb{R})$ and $\phi \in C^1(\mathbb{R})$. Then, the problem (6.8)–(6.9) admits a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ given by:*

$$u(x, t) = \frac{1}{2} [\psi(x - ct) + \psi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(s) ds + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau) d\xi d\tau. \quad (6.10)$$

Remark 9. *The formula (6.10) is called d'Alembert's formula for the non homogeneous wave equation in one dimension.*

Proof. We introduce the following change of variables:

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct, \end{cases}$$

Then, the PDE (6.8) can be rewritten in the form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = -\frac{1}{4c^2} F \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right). \quad (6.11)$$

Integrating with respect to ξ , we get

$$\begin{aligned} \frac{\partial u}{\partial \eta} \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) &= \frac{\partial u}{\partial \eta} \left(\frac{\bar{\xi} + \eta}{2}, \frac{\bar{\xi} - \eta}{2c} \right) \Big|_{\bar{\xi}=\eta} + \int_{\eta}^{\xi} \frac{\partial^2 u}{\partial \xi \partial \eta} \left(\frac{\bar{\xi} + \eta}{2}, \frac{\bar{\xi} - \eta}{2c} \right) d\bar{\xi} \\ &= \frac{1}{2} \frac{\partial u}{\partial x}(\eta, 0) - \frac{1}{2c} \frac{\partial u}{\partial t}(\eta, 0) - \frac{1}{4c^2} \int_{\eta}^{\xi} F \left(\frac{\bar{\xi} + \eta}{2}, \frac{\bar{\xi} - \eta}{2c} \right) d\bar{\xi}. \end{aligned}$$

Integrating this equation over the interval (η, ξ) gives:

$$\begin{aligned}
 u(\xi, 0) - u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right) &= \underbrace{\frac{1}{2} \int_{\eta}^{\xi} \frac{\partial u}{\partial x}(\bar{\eta}, 0) d\bar{\eta}}_{I_1} - \frac{1}{2c} \int_{\eta}^{\xi} \frac{\partial u}{\partial t}(\bar{\eta}, 0) d\bar{\eta} \\
 &\quad - \underbrace{\frac{1}{4c^2} \int_{\eta}^{\xi} \int_{\bar{\eta}}^{\xi} F\left(\frac{\bar{\xi} + \bar{\eta}}{2}, \frac{\bar{\xi} - \bar{\eta}}{2c}\right) d\bar{\xi} d\bar{\eta}}_{I_2}.
 \end{aligned} \tag{6.12}$$

Now, in I_1 , we have

$$\frac{1}{2} \int_{\eta}^{\xi} \frac{\partial u}{\partial x}(\bar{\eta}, 0) d\bar{\eta} = \frac{1}{2} (u(\xi, 0) - u(\eta, 0)).$$

For I_2 , we set:

$$\begin{cases} \bar{\xi} = \bar{x} + c\bar{t}, \\ \bar{\eta} = \bar{x} - c\bar{t}. \end{cases}$$

The integration domain $\eta \leq \bar{\eta} \leq \bar{\xi} \leq \xi$ becomes: $\eta \leq \bar{x} - c\bar{t} \leq \bar{x} + c\bar{t} \leq \xi$, or equivalently:

$$\eta + c\bar{t} \leq \bar{x} \leq \xi - c\bar{t}, \quad \text{and} \quad 0 \leq \bar{t} \leq \frac{1}{2c}(\xi - \eta).$$

The Jacobian determinant of the transformation is:

$$\begin{vmatrix} \frac{\partial \bar{\xi}}{\partial \bar{x}} & \frac{\partial \bar{\xi}}{\partial \bar{t}} \\ \frac{\partial \bar{\eta}}{\partial \bar{x}} & \frac{\partial \bar{\eta}}{\partial \bar{t}} \end{vmatrix} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c.$$

Hence,

$$I_2 = \frac{1}{2c} \int_0^{(\xi-\eta)/2c} \int_{\eta+c\bar{t}}^{\xi-c\bar{t}} F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}.$$

Substituting I_1 and I_2 into equation (6.12), we obtain:

$$\begin{aligned}
 u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right) &= \frac{1}{2} (u(\xi, 0) + u(\eta, 0)) + \frac{1}{2c} \int_{\eta}^{\xi} \frac{\partial u}{\partial t}(\bar{x}, 0) d\bar{x} \\
 &\quad + \frac{1}{2c} \int_0^{(\xi-\eta)/2c} \int_{\eta+c\bar{t}}^{\xi-c\bar{t}} F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}.
 \end{aligned}$$

Now, noting that $\xi = x + ct$ and $\eta = x - ct$, and using the initial conditions (6.9), we recover the formula (6.10). \square

Example 25. Using d'Alembert's formula, determine the solution of the equation (6.8) with $F(x, t) = x$, $c = 1$, and the following initial conditions:

$$u(x, 0) = u_t(x, 0) = 0.$$

Solution: According to the formula (6.10), we have:

$$u(x, t) = \frac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi \, d\tau.$$

We calculate the first integral:

$$\int_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2} [\xi^2]_{x-(t-\tau)}^{x+(t-\tau)} = \frac{1}{2} [(x + (t - \tau))^2 - (x - (t - \tau))^2].$$

Then

$$\int_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2} [(x^2 + 2x(t - \tau) + (t - \tau)^2) - (x^2 - 2x(t - \tau) + (t - \tau)^2)].$$

The terms in x^2 and $(t - \tau)^2$ cancel out, and we obtain

$$\int_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2} [4x(t - \tau)].$$

Thus, we have:

$$u(x, t) = \frac{1}{2} \int_0^t 4x(t - \tau) \, d\tau = 2x \int_0^t (t - \tau) \, d\tau.$$

The integral is straightforward to calculate:

$$\int_0^t (t - \tau) \, d\tau = \frac{t^2}{2}.$$

Therefore, the solution is:

$$u(x, t) = 2x \times \frac{t^2}{2} = xt^2.$$

Thus, the solution to the equation (6.8) is:

$$u(x, t) = \frac{xt^2}{2}.$$

6.3 Exercises

Exercise 6.1. Solve

$$\begin{aligned} y_{tt} &= 9y_{xx} && \text{for } 0 < x < 1 \text{ and } t > 0, \\ y(0, t) &= y(1, t) = 0 && \text{for } t > 0, \\ y(x, 0) &= \sin(3\pi x) + \frac{1}{4}\sin(6\pi x) && \text{for } 0 < x < 1, \\ y_t(x, 0) &= 0 && \text{for } 0 < x < 1. \end{aligned}$$

Exercise 6.2. Solve

$$\begin{aligned} y_{tt} &= 4y_{xx} && \text{for } 0 < x < 1 \text{ and } t > 0, \\ y(0, t) &= y(1, t) = 0 && \text{for } t > 0, \\ y(x, 0) &= \sin(3\pi x) + \frac{1}{4}\sin(6\pi x) && \text{for } 0 < x < 1, \\ y_t(x, 0) &= \sin(9\pi x) && \text{for } 0 < x < 1. \end{aligned}$$

Exercise 6.3. Let $u(x, t)$ be thrice continuously differentiable solution of the initial value problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && (|x| < \infty, t > 0) \\ u(x, 0) &= 0, \quad u_t(x, 0) = g(x) && (|x| < \infty) \end{aligned}$$

where g is twice continuously differentiable.

Set $v(x, t) = u_t(x, t)$, show that $v(x, t)$ satisfies the same differentiable equation and the initial condition $v(x, 0) = g(x)$, $v_t(x, 0) = 0$.

Exercise 6.4. Find the solution of the initial value problem:

$$\begin{aligned} u_{tt} - u_{xx} - au_t - au_x &= 0 && (|x| < \infty, t > 0) \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x). \end{aligned}$$

Hint: let $u(x, t) = v(x, t) \exp(ax + bt)$ and choose the constants a and b in such a way that the

equation reduces to the wave equation.

Exercise 6.5. Let $u(x, t, \tau)$ be the solution of the initial value problem:

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0, & (|x| < \infty) \\u(x, 0, \tau) &= 0, \quad u_t(x, 0, \tau) = F(x, \tau).\end{aligned}$$

Set

$$v(x, t) = \int_0^t u(x, t - \tau, \tau) d\tau.$$

(a). Show that $v(x, t)$ satisfies the non homogeneous equation $v_{tt} - c^2 v_{xx} = F(x, t)$ and the initial condition:

$$v(x, 0) = 0 \qquad v_t(x, 0) = 0$$

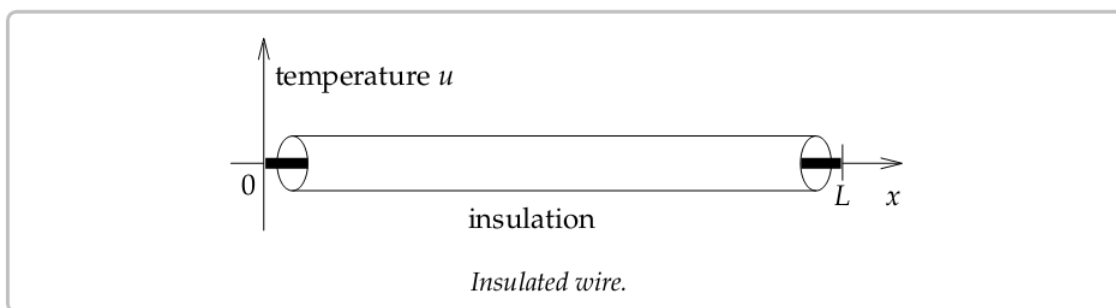
(b). Express $u(x, t, \tau)$ by using d'Alembert formula.

Chapter 7

Heat equation

7.1 Introduction

Consider a wire (or a thin metal rod) of length L insulated along its length except at the endpoints. Let x denote the position along the wire and let t denote time. See the following image:



Let $u(x, t)$ denote the temperature at point x at time t . The equation governing this setup is the so-called *one-dimensional heat equation*:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

where $k > 0$ is a constant (the *thermal conductivity* of the material). That is, the change in heat with respect to time at some point is proportional to the second derivative of the heat in the x direction—along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down and the second x derivative is negative), then heat should flow away from the maximum and so the t derivative should also be negative. Similarly at a minimum, heat wants to flow in.

We generally use a more convenient notation for partial derivatives. We write u_t instead of $\frac{\partial u}{\partial t}$, and we write u_{xx} instead of $\frac{\partial^2 u}{\partial x^2}$. With this notation the heat equation becomes

$$u_t = ku_{xx}.$$

The region in which we will solve the heat equation is given by

$$0 < x < L \quad \text{and} \quad t > 0.$$

We must also have some side conditions on the boundaries of that region. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. If the ends of the wire are kept at temperature 0, then the conditions are

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \quad \text{for } t > 0.$$

If, on the other hand, the ends are insulated, the conditions are

$$u_x(0, t) = 0 \quad \text{and} \quad u_x(L, t) = 0 \quad \text{for } t > 0.$$

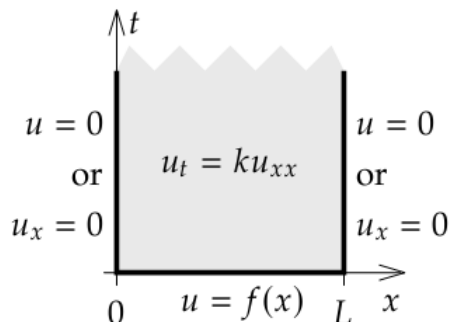
Let us see why that is so. If u_x is positive at some point x_0 , then at a particular time, u is smaller to the left of x_0 and higher to the right of x_0 . Heat is flowing from high heat to low heat, that is, to the left. On the other hand, if u_x is negative, then heat is again flowing from high heat to low heat, that is, to the right. So when u_x is zero, we are at a point where heat is not flowing in either direction. In other words, $u_x(0, t) = 0$ means no heat is flowing in or out of the wire at the point $x = 0$.

We have two conditions along the x -axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e. u or a derivative of u is set to zero).

We also need an initial condition—the temperature distribution at time $t = 0$. That is,

$$u(x, 0) = f(x) \quad \text{for } 0 < x < L,$$

for some known function $f(x)$. This initial condition is not a homogeneous side condition.



7.2 Separation of variables

The method of *separation of variables* is to try to find solutions that are products of functions of one variable. For the heat equation, we try to find solutions of the form

$$u(x, t) = X(x)T(t).$$

That the desired particular solution we are looking for is of this form is too much to hope for. What is perfectly reasonable to ask, however, is to find enough solutions of the form $u(x, t) = X(x)T(t)$ using this procedure so that the desired solution to the PDE is somehow constructed from these building blocks by the use of superposition.

Let us try to solve the heat equation problem

$$u_t = k u_{xx}, \quad \text{with } u(0, t) = 0, \quad u(L, t) = 0, \quad \text{and } u(x, 0) = f(x).$$

We guess $u(x, t) = X(x)T(t)$. We will try to make this guess satisfy the differential equation, $u_t = k u_{xx}$, and the homogeneous side conditions, $u(0, t) = 0$ and $u(L, t) = 0$. Then, as superposition preserves the differential equation and the homogeneous side conditions, we will try to build up a solution from these building blocks to solve the nonhomogeneous initial condition $u(x, 0) = f(x)$.

First, we plug $u(x, t) = X(x)T(t)$ into the heat equation to obtain

$$X(x)T'(t) = kX''(x)T(t).$$

We rewrite as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

This equation must hold for all x and all t . But the left-hand side does not depend on x and

the right-hand side does not depend on t . Hence, each side must be a constant. Let us call this constant $-\lambda$ (the minus sign is for convenience later). We obtain the two equations

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

In other words,

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda kT(t) &= 0. \end{aligned}$$

The boundary condition $u(0, t) = 0$ implies $X(0)T(t) = 0$. We are looking for a nontrivial solution, and so we can assume that $T(t)$ is not identically zero. Hence $X(0) = 0$. Similarly, $u(L, t) = 0$ implies $X(L) = 0$. We are looking for nontrivial solutions X of the eigenvalue problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We have previously found that the only eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$, for integers $n \geq 1$, where eigenfunctions are $\sin\left(\frac{n\pi}{L}x\right)$. Hence, let us pick the solutions

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding T_n must satisfy the equation

$$T_n'(t) + \frac{n^2\pi^2}{L^2}kT_n(t) = 0.$$

This is one of our fundamental equations, and the solution is an exponential:

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt},$$

where we picked the particular solution where conveniently $T_n(0) = 1$. Our building-block solutions are

$$u_n(x, t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

We note that $u_n(x, 0) = \sin\left(\frac{n\pi}{L}x\right)$. We write $f(x)$ as the sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the odd periodic extension of $f(x)$. We used the sine series

as it corresponds to the eigenvalue problem for $X(x)$ above. Finally, we use superposition to write the solution as

$$u(x, t) = \sum_{n=1}^{\infty} b_n u_n(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

Why does this solution work? First note that it is a solution to the heat equation by superposition. It satisfies $u(0, t) = 0$ and $u(L, t) = 0$, because $x = 0$ or $x = L$ makes all the sines vanish. Finally, plugging in $t = 0$, we notice that $T_n(0) = 1$, and so

$$u(x, 0) = \sum_{n=1}^{\infty} b_n u_n(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

7.2.1 Insulated ends

Now suppose the ends of the wire are insulated. In this case, we are solving the problem

$$u_t = ku_{xx} \quad \text{with} \quad u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

Yet again we try a solution of the form $u(x, t) = X(x)T(t)$. By the same procedure as before, we plug into the heat equation and arrive at the following two equations:

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda k T(t) &= 0. \end{aligned}$$

At this point, the story changes slightly. The boundary condition $u_x(0, t) = 0$ implies $X'(0)T(t) = 0$. Hence $X'(0) = 0$. Similarly, $u_x(L, t) = 0$ implies $X'(L) = 0$. We want nontrivial solutions X of the eigenvalue problem $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. We previously found that the only eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$, for integers $n \geq 0$, where eigenfunctions are $\cos\left(\frac{n\pi}{L}x\right)$ (we include the constant eigenfunction). We pick the solutions

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \quad \text{and} \quad X_0(x) = 1.$$

The corresponding T_n must satisfy the equation

$$T'_n(t) + \frac{n^2\pi^2}{L^2}kT_n(t) = 0.$$

For $n \geq 1$, as before,

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}.$$

For $n = 0$, we have $T_0'(t) = 0$ and hence $T_0(t) = 1$. Our building-block solutions are

$$u_n(x, t) = X_n(x)T_n(t) = \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

and

$$u_0(x, t) = 1.$$

We note that $u_n(x, 0) = \cos\left(\frac{n\pi}{L}x\right)$. We write f using the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the even periodic extension of $f(x)$.

We use superposition to write the solution as

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n u_n(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

7.3 Exercises

Exercise 7.1. Verify that each of the given functions satisfies the heat equation for $0 < x < \pi$ and the accompanying initial and boundary conditions.

- (a) $u(x, t) = e^{-kt} \sin x$; $u(x, 0) = \sin x$, $u(0, t) = u(\pi, t) = 0$.
- (b) $u(x, t) = e^{-kt} \cos x$; $u(x, 0) = \cos x$, $u_x(0, t) = u_x(\pi, t) = 0$.
- (c) $u(x, t) = \frac{1}{2} + \frac{1}{2}e^{-4kt} \cos 2x$; $u(x, 0) = \cos^2 x$, $u_x(0, t) = u_x(\pi, t) = 0$

Exercise 7.2. Consider the initial boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= F(x, t) && (0 < x < L, t > 0) \\ u(x, 0) &= F(x) && (0 \leq x \leq L) \\ u(0, t) &= a(t), \quad u(L, t) = b(t) && (t \geq 0) \end{aligned}$$

- (a) Determine a function of the form $\phi(x, t) = A(t) + xB(t)$ such that $\phi(0, t) = a(t)$ and $\phi(L, t) = b(t)$.
- (b) By introducing a new function v , defined by $v = u - \phi$, reduce problem (a) to one in which the boundary conditions are homogeneous.

- (c) Using the method of separation of variables find the general solution to original problem.

Exercise 7.3. Consider the initial boundary value problem

$$u_t - ku_{xx} = F(x, t) \quad (0 < x < L, t > 0)$$

with the boundary conditions $u_x(0, t) = a(t)$, $u_x(L, t) = b(t)$.

(a) Determine a function ϕ such that $\phi_x(0, t) = a(t)$ and $\phi_x(L, t) = b(t)$ and thus reduce the problem to one with homogeneous boundary conditions.

(a) Using the method of separation of variables find the general solution to original problem.

Exercise 7.4. Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= Ae^{-ax} && (a > 0, 0 < x < L, t > 0) \\ u(x, 0) &= F(x) && (0 \leq x \leq L) \\ u(0, t) &= 0, \quad u(L, t) = 0 && (t \geq 0) \end{aligned}$$

where A is a constant. Introduce a new function v , defined by $v(x, t) = u(x, t) + \phi(x)$, where ϕ is a function of only the variable x .

(a) Determine ϕ so that v satisfies $v_t - kv_{xx} = 0$.

(b) Determine ϕ so that v also satisfies $v(0, t) = 0$ and $v(L, t) = 0$, and thus reduce the given problem to one with homogeneous differential equation and boundary conditions.

(c) Using the method of separation of variables find the general solution to original problem.

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