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Course Notes
Partial Differential Equations

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Chapter 1

Elliptic equation

1.1 Introduction

The partial differential equation of elliptic type naturally appears in physical problems such as: *electrostatics*, *fluid flow studies*, *studies of elastic membranes*, etc. A simple example of an elliptic-type equation is the Poisson equation. More specifically, an equation written in the following form:

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \text{in} \quad \Omega, \quad \text{with} \quad \Delta = \frac{\partial^2}{\partial x_i^2}$$

where Ω is a bounded open subset of $\mathbb{R}^n$.

1.2 Laplace's equation

Let Ω be a bounded regular open set in $\mathbb{R}^n$ with $n \geq 1$ and $\partial\Omega$ its boundary. If the unknown $u(\mathbf{x})$, $\big(\mathbf{x} = (x_1, \dots, x_n) \in \Omega\big)$, satisfies the following equation in Ω:

$$\Delta u(\mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where Δ denotes the partial differential operator, that is:

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Then, equation (1.1) is called Laplace's equation in Ω.

Remark 1. *The non-homogeneous Laplace equation*

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \qquad (\mathcal{P})$$

where f is given, is called the Poisson equation.

Remark 2. *From a physical point of view, we encounter the Poisson equation and the Laplace equation, for example, in the following problems:*
** **In electrostatics:** According to Maxwell's equations*[1]*, we have* $\mathrm{rot}\ (\mathbf{E}) = 0$ *and* $\mathrm{div}\ (\mathbf{E}) = 4\pi\rho$*, where ρ is the charge density. The first equation implies that* $\boldsymbol{E} = \mathrm{grad}\ (\phi)$ *(for a scalar function ϕ called the electric potential). Then,*

$$\Delta\phi = \mathrm{div}\big(\mathrm{grad}\,(\phi)\big) = \mathrm{div}\ \boldsymbol{E} = -4\pi\rho,$$

and for $f = -4\pi\rho$, this equation is a Poisson equation.
** **Study of fluid flow:** Suppose the flow is irrotational, such that* $\mathrm{rot}\ \mathrm{v} = 0$*, where* $\mathrm{v} = \mathrm{v}(x, y, z)$ *is the velocity at the position (x, y, z), assuming it is independent of time. If the fluid is incompressible (e.g., water), then* $\mathrm{div}\ \mathrm{v} = 0$*, which gives* $\mathrm{v} = -\mathrm{grad}\ \phi$ *for some ϕ, (ϕ is called the velocity potential), and the following Laplace equation:* $\Delta\phi = -\mathrm{div}\ \mathrm{v} = 0$.

1.3 Harmonic functions

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $\partial\Omega$ its boundary. In the following, we denote by $\mathbf{x} = (x_1, ..., x_n)$.

Definition 1. *Let $u \in C^2(\Omega)$ be a real-valued function. We say that u is harmonic if:*

$$\Delta u(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(\mathbf{x}) = 0.$$

Example 1. *Show that the function u defined by*

$$u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$

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[1] Maxwell's equations for an electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ are:

$$\begin{cases} \mathbf{E}_t = \mathrm{rot}\ (\mathbf{H}) \\ \mathbf{H}_t = -\mathrm{rot}\ (\mathbf{E}) \\ \mathrm{div}\ (\mathbf{H}) = \mathrm{div}\ (\mathbf{E}) = 0 \end{cases}$$

is a harmonic function on the disk $D = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$.

1.4 Dirichlet problem for Laplace's equation in the rectangle

Consider the Dirichlet problem for Laplace's equation in a rectangle R (see figure 1.1).

The corresponding mathematical model is written as: find $u : (x, y) \mapsto u(x, y)$ such that

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad 0 < x < a, \ 0 < y < b \tag{1.2}$$

with the boundary conditions

$$\begin{cases} u(0,y) = 0, & u(a,y) = 0, \qquad 0 \leq y \leq b \\ u(x,0) = 0, & u(x,b) = f(x), \quad 0 \leq x \leq a, \end{cases} \tag{1.3}$$

where f is given.

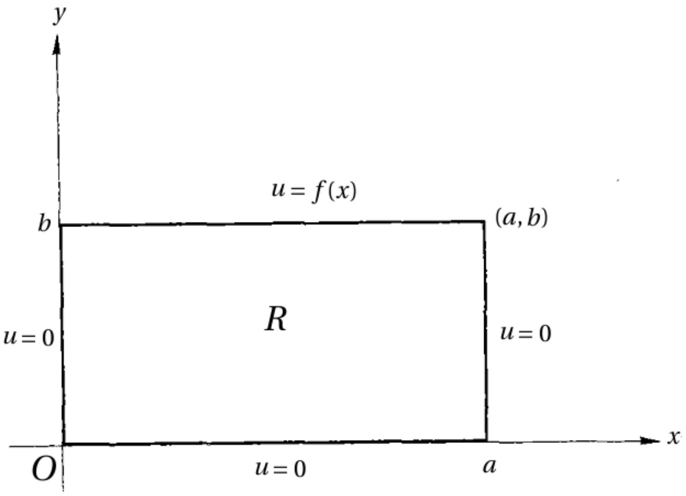

Figure 1.1: The Dirichlet problem in a rectangle R.

According to the method of separation of variables, we assume that any non-trivial solution has the form:

$$u(x, y) = X(x)Y(y) \tag{1.4}$$

which satisfies the three homogeneous conditions in (1.3). We have dropped the non-homogeneous condition $u(x, b) = f(x)$.

Substituting this solution into the PDE (1.2) and separating the variables, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0 \tag{1.5}$$

where X'' is the second derivative of X with respect to x, and Y'' is the second derivative of Y with respect to y. Dividing the equation (1.5) by $X(x)Y(y)$, we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \Longrightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \tag{1.6}$$

In (1.6), the left-hand term is a function of x only, while the right-hand term is a function of y only. Therefore, these two expressions must be constant, and we can write

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda, \quad \lambda \in \mathbb{R}. \tag{1.7}$$

According to the boundary conditions (1.3), the solution u satisfies the three homogeneous conditions, and we obtain:

$$\begin{cases} u(x,0) = 0 \Longrightarrow X(x)Y(0) = 0, \quad x \in [0,a] \Longrightarrow Y(0) = 0, \\ u(0,y) = 0 \Longrightarrow X(0)Y(y) = 0, \quad y \in [0,b] \Longrightarrow X(0) = 0, \\ u(a,y) = 0 \Longrightarrow X(a)Y(y) = 0, \quad y \in [0,b] \Longrightarrow X(a) = 0. \end{cases} \tag{1.8}$$

In summary, we need to study the following problems:

$$(\mathcal{P}_1) \begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < a, \\ X(0) = 0, \ X(a) = 0 \end{cases}$$

and

$$(\mathcal{P}_2) \begin{cases} Y''(y) - \lambda Y(y) = 0, \quad 0 < y < b, \\ Y(0) = 0. \end{cases}$$

Solution to the problem $(\mathcal{P}_1)$

A non-trivial solution of $(\mathcal{P}_1)$ is called an eigenfunction for the eigenvalue λ. We distinguish three possible cases for λ.

Case 1: if $\lambda = -k^2 < 0$, with k being a non-zero real number. Then, the general solution to problem $(\mathcal{P}_1)$ is of the form:

$$X(x) = c_1 e^{-kx} + c_2 e^{kx}$$

where c_1 and c_2 are positive constants. According to the conditions $X(0) = 0,\ X(a) = 0$, we have the system

$$(S)\ \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{-ka} + c_2 e^{ka} = 0. \end{cases}$$

The system (S) has only one solution, which is the trivial solution $(c_1, c_2) = (0, 0)$. In fact, the determinant of the system (S) is

$$\begin{vmatrix} 1 & 1 \\ e^{-ka} & e^{ka} \end{vmatrix} = e^{ka} - e^{-ka} = \frac{e^{2ka} - 1}{e^{ka}} \neq 0.$$

Otherwise,

$$e^{2ka} - 1 = 0 \Longrightarrow e^{2ka} = 1 \Longrightarrow 2ka = 0.$$

But this is absurd because k and a are not zero. Thus, we can conclude that

$$\begin{pmatrix} 1 & 1 \\ e^{-ka} & e^{ka} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because the matrix $\begin{pmatrix} 1 & 1 \\ e^{-ka} & e^{ka} \end{pmatrix}$ is invertible. Therefore, in the case where $\lambda = -k^2 < 0$, we have $X(x) \equiv 0$ and $u(x, t) \equiv 0$ for all $0 < x < a$. Thus, we must exclude the case $\lambda < 0$.

Case 2: if $\lambda = 0$ we have $X(x) = c_1 x + c_2$ where c_1 and c_2 are arbitrary real numbers. However, according to the conditions $X(a) = 0$ and $X(0) = 0$, we have

$$\begin{cases} c_2 = 0 \\ c_1 a + c_2 = 0 \end{cases} \Longrightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases} \quad \text{because } a \neq 0.$$

Thus, in the case where $\lambda = 0$, we have $X(x) \equiv 0$ and $u(x, t) \equiv 0$ for all $0 < x < a$. Therefore, we must also exclude the case $\lambda = 0$.

Case 3: if $\lambda > 0$, the general solution to problem $(\mathcal{P}_1)$ is of the form

$$X(x) = c_1 \sin\big(\sqrt{\lambda} x\big) + c_2 \cos\big(\sqrt{\lambda} x\big)$$

where c_1 and c_2 are arbitrary real numbers. According to the condition $X(0) = 0$, we obtain c_2. Thus, a non-trivial solution is of the form $X(x) = c_1 \sin\big(\sqrt{\lambda} x\big)$ with $c_1 \neq 0$.

Also, according to the condition $X(a) = 0$, we have

$$X(a) = c_1 \sin\big(\sqrt{\lambda} a\big) = 0 \Longrightarrow \sin\big(\sqrt{\lambda} a\big) = 0.$$

Thus,

$$\sqrt{\lambda} a = n\pi, \quad n \in \mathbb{Z}.$$

Therefore, we conclude that the eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Thus, the eigenfunctions corresponding to these eigenvalues are

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right). \tag{1.9}$$

Now we move on to problem $(\mathcal{P}_2)$. For $n \geq 1$, $n \in \mathbb{N}$ in the case where $\lambda = \lambda_n = \left(\frac{n\pi}{a}\right)^2$, the general solution is of the form:

$$Y_n(y) = Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}},$$

where C and D are arbitrary constants. According to the condition $Y_n(0) = 0$, we have $C = -D$, thus

$$Y_n(y) = C\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right), \quad \text{with} \quad n \in \mathbb{N}^*.$$

We also have the identity $\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$, for all $\theta \in \mathbb{R}$, so

$$Y_n(y) = 2C \sinh\left(\frac{n\pi y}{a}\right), \quad n = 1, 2, \ldots \tag{1.10}$$

Finally, according to (1.9) and (1.10), the general solution of the PDE (1.2) with the boundary conditions (1.3) is written in the form:

$$\begin{aligned} u_n(x, y) = X_n(x) Y_n(y) &= c_1^n \sin\left(\frac{n\pi x}{a}\right) 2C \sinh\left(\frac{n\pi y}{a}\right), \quad n \in \mathbb{N}^* \\ &= C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), \quad n \in \mathbb{N}^* \end{aligned} \tag{1.11}$$

with $C_n = 2c_1^n C$. Note that, according to **the principle superposition** [2]

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \tag{1.13}$$

is also a solution of the problem (1.2)–(1.3), where C_n are constants to be determined.

If we take $y = b$ in (1.13), using the condition $u(x,b) = f(x)$, we obtain:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) \\ &= \sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right) \\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right), \quad x \in [0,a] \end{aligned}$$

where

$$b_n = C_n \sinh\left(\frac{n\pi b}{a}\right), \quad (n = 1, 2, ...). \tag{1.14}$$

Thus, the expression $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)$, for all $x \in [0,a]$, is the odd Fourier series of the function f on $[0,a]$, so

$$b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx, \quad (n = 1, 2, ...).$$

Finally, according to (1.14), we have: $C_n = \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} b_n$ and substituting C_n into (1.13) gives:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} b_n \sin\left(\frac{n\pi x}{a}\right) \tag{1.15}$$

which is a solution of the PDE (1.2) with the boundary conditions (1.3).

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[2]

Theorem 1. (*Principle Superposition*). *For any $k \in \mathbb{N}$, if u_1, u_2, ..., u_k are solutions of a linear and homogeneous partial differential equation of order m given by:*

$$F\left(x, u, D^1(u), D^2(u), ..., D^m(u)\right) = 0 \tag{1.12}$$

for all $x = (x_1, x_2, ..., x_n) \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$ and $D^{\alpha}(u) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$ with $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + ... + \alpha_n$, then $U = c_1 u_1 + c_2 u_2 + ... + c_k u_k$ is also a solution of (1.12) for any constants c_1, c_2, ..., c_k. That is:

$$F\left(x, u, D^1(U), D^2(U), ..., D^m(U)\right) = 0$$

for all $x \in \Omega$.

Example 2. *Determine the solution of the following Laplace equation:*

$$u_{xx}(x,y)+u_{yy}(x,y)=0, \quad 0<x<\pi, \quad 0<y<\pi \tag{1.16}$$

with the following boundary conditions:

$$\begin{cases} u(0,y)=0, \quad u(\pi,y)=0, \quad 0\le y\le\pi, \\ u(x,0)=0, \quad u(x,\pi)=\sin^3(x), \quad 0\le x\le\pi. \end{cases} \tag{1.17}$$

Solution: From (1.15), the solution of the problem (1.16)–(1.17) is given by:

$$u(x,y)=\sum_{n=1}^{\infty}\frac{\sinh(ny)}{\sinh(n\pi)}b_n\sin(nx),$$

with

$$b_n=\frac{2}{\pi}\int_0^{\pi}\sin^3(x)\sin(nx)dx, \quad (n=1,2,...).$$

Now, we have $\sin^3(x)=\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)$, so

$$b_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)\right)\sin(nx)dx, \quad (n=1,2,...).$$

For $n=1$:

$$b_1=\frac{3}{2\pi}\int_0^{\pi}\sin(x)\sin(x)dx-\frac{1}{2\pi}\int_0^{\pi}\sin(3x)\sin(x)dx=\frac{3}{4}.$$

For $n=2$:

$$b_2=\frac{3}{2\pi}\int_0^{\pi}\sin(x)\sin(2x)dx-\frac{1}{2\pi}\int_0^{\pi}\sin(3x)\sin(2x)dx=0.$$

For $n=3$:

$$b_3=\frac{3}{2\pi}\int_0^{\pi}\sin(x)\sin(3x)dx-\frac{1}{2\pi}\int_0^{\pi}\sin(3x)\sin(3x)dx=-\frac{1}{4}.$$

And for $n\ge4$, we have $b_n=0$. Therefore, the solution to the problem (1.16)–(1.17) is:

$$u(x,y)=\frac{3\sinh(y)}{4\sinh(\pi)}\sin(x)-\frac{\sinh(3y)}{4\sinh(3\pi)}\sin(3x).$$

1.5 Dirichlet problem for Laplace's equation in the Circle

Let $D = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \leq a^2\}$ be a disk of radius a centered at the origin. We denote by $C : x^2+y^2 = a^2$ the circle centered at $(0,0)$ with radius a, which is the boundary of D. We are looking for a harmonic function in D that satisfies $u = f$ on C, as shown in the following figure:

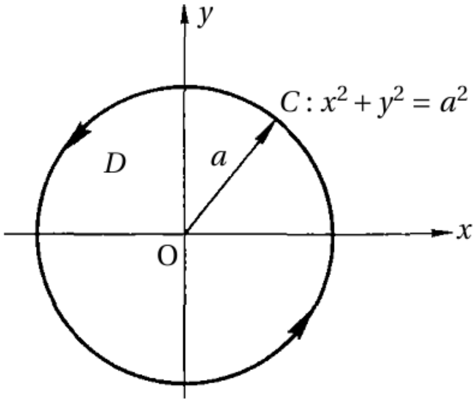


Figure 1.2: The Dirichlet problem in a disk D.

The problem can therefore be written as: find a function u such that

$$(P) : \begin{cases} \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 0, & \text{in} \quad D, \\ u(x,y) = f(x,y), & \text{on} \quad C, \end{cases}$$

where f is given.

We cannot apply the method of separation of variables because the domain D is not the product of two intervals. Therefore, we can try to look for the solution in the form:

$$u(x,y) = w(r,\theta) \tag{1.18}$$

with

$$\begin{cases} x = r\cos(\theta), & 0 < r \leq a, \\ y = r\sin(\theta), & 0 \leq \theta \leq 2\pi. \end{cases} \tag{1.19}$$

Lemma 1. *Let u be a function of class $C^2(D)$. Then, using (1.18) and (1.19), the problem (P) is equivalent to*

$$(P') : \begin{cases} \Delta w(r,\theta) = w_{rr}(r,\theta) + \frac{1}{r} w_r(r,\theta) + \frac{1}{r^2} w_{\theta\theta}(r,\theta) = 0, \quad 0 < r \leq a, \\ w(a,\theta) = h(\theta), \quad 0 \leq \theta \leq 2\pi, \end{cases}$$

with $w(a,\theta) = h(\theta) = f\big(x(a,\theta),\, y(a,\theta)\big)$.

1.5.1 Solution of the problem (P') by using the method of separation of variables

We begin by seeking non-trivial solutions of the problem (P') in separated variables of the form:

$$w(r,\,\theta) = F(r)G(\theta). \tag{1.20}$$

By substituting (1.20) into (P'), we obtain

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0, \tag{1.21}$$

where F'' is the second derivative of F with respect to r, and G'' is the second derivative of G with respect to θ. Dividing equation (1.21) by $F(r)G(\theta)$, we get

$$\frac{F''(r)}{F(r)} + \frac{1}{r}\frac{F'(r)}{F(r)} + \frac{1}{r^2}\frac{G''(\theta)}{G(\theta)} = 0,$$

and therefore

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}. \tag{1.22}$$

The left-hand side of (1.22) depends only on the variable r, and the right-hand side depends only on θ. Hence, there exists a real constant λ such that:

$$r^2\frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = \lambda. \tag{1.23}$$

We obtain two ordinary differential equations:

$$r^2F''(r) + rF'(r) - \lambda F(r) = 0$$

and

$$G''(\theta) + \lambda G(\theta) = 0$$

for the functions F and G, respectively. However, the function w must be 2π-periodic with respect to θ, that is,

$$w(r,\,\theta + 2\pi) = w(r,\,\theta).$$

Thus, the function G must be periodic with period 2π. Consequently, we impose periodic boundary conditions on G, that is,

$$\begin{cases} G(0) - G(2\pi) = 0, \\ G'(0) - G'(2\pi) = 0. \end{cases}$$

In summary, we need to study the following problems:

$$(P_1') \begin{cases} G''(\theta) + \lambda G(\theta) = 0, \;\; 0 \le \theta \le 2\pi, \\ G(0) - G(2\pi) = 0, \\ G'(0) - G'(2\pi) = 0, \end{cases}$$

and

$$(P_2') \begin{cases} r^2 F''(r) + rF'(r) - \lambda F(r) = 0, \\ 0 < r \le a. \end{cases}$$

Solution of the problem (P_1').

As is customary, different cases arise depending on the sign of λ.

Case 1: If $\lambda < 0$, the general solution of the problem (P_1') is given by:

$$G(\theta) = c_1 \cosh\big(\sqrt{-\lambda}\,\theta\big) + c_2 \sinh\big(\sqrt{-\lambda}\,\theta\big),$$

where c_1 and c_2 are arbitrary constants. However, since $G(0) - G(2\pi) = 0$ and $G'(0) - G'(2\pi) = 0$, we get $c_1 = c_2 = 0$, which is impossible. Thus, the case $\lambda < 0$ must be rejected.

Case 2: If $\lambda = 0$, the general solution of the problem (P_1') is written as $G(\theta) = c_1\theta + c_2$, where c_1 and c_2 are arbitrary constants. According to the conditions $G(0) - G(2\pi) = 0$ and $G'(0) - G'(2\pi) = 0$, we obtain $c_1 = c_2 = 0$. Therefore, the case $\lambda = 0$ must also be rejected.

Case 3: If $\lambda > 0$, the general solution of the problem (P_1') is

$$G(\theta) = c_1 \cos\big(\sqrt{\lambda}\,\theta\big) + c_2 \sin\big(\sqrt{\lambda}\,\theta\big),$$

where c_1 and c_2 are arbitrary constants. The conditions $G(0) = G(2\pi)$ and $G'(0) = G'(2\pi)$ give us

$$(s) : \begin{cases} c_1 = c_1 \cos\big(2\pi\sqrt{\lambda}\big) + c_2 \sin\big(2\pi\sqrt{\lambda}\big) \\ c_2\sqrt{\lambda} = \sqrt{\lambda}\Big[c_2 \cos\big(2\pi\sqrt{\lambda}\big) - c_1 \sin\big(2\pi\sqrt{\lambda}\big)\Big] \end{cases}$$

which is equivalent to

$$(s):\ \begin{cases} c_1\Big(\cos\big(2\pi\sqrt{\lambda}\big)-1\Big)+c_2\sin\big(2\pi\sqrt{\lambda}\big)=0\\ -c_1\sin\big(2\pi\sqrt{\lambda}\big)+c_2\Big(\cos\big(2\pi\sqrt{\lambda}\big)-1\Big)=0. \end{cases}$$

If the determinant of (s) is non-zero, then $c_1 = c_2 = 0$, and $G \equiv 0$. Non-trivial solutions are obtained if $\det(s) = 0$. We then have

$$\begin{aligned} \det(s)=0 &\Longleftrightarrow \begin{vmatrix} \cos\big(2\pi\sqrt{\lambda}\big)-1 & \sin\big(2\pi\sqrt{\lambda}\big)\\ -\sin\big(2\pi\sqrt{\lambda}\big) & \cos\big(2\pi\sqrt{\lambda}\big)-1 \end{vmatrix}=0\\ &\Longleftrightarrow \Big(\cos\big(2\pi\sqrt{\lambda}\big)-1\Big)+\sin^2\big(2\pi\sqrt{\lambda}\big)=0\\ &\Longleftrightarrow \cos^2\big(2\pi\sqrt{\lambda}\big)-2\cos\big(2\pi\sqrt{\lambda}\big)+1+\sin^2\big(2\pi\sqrt{\lambda}\big)=0\\ &\Longleftrightarrow -2\cos\big(2\pi\sqrt{\lambda}\big)+2=0\\ &\Longleftrightarrow \cos\big(2\pi\sqrt{\lambda}\big)=1 \Longleftrightarrow 2\pi\sqrt{\lambda}=2n\pi,\quad n\in\mathbb{N},\\ &\qquad\qquad\qquad\qquad\ \Longleftrightarrow \lambda_n=n^2,\quad n\in\mathbb{N}. \end{aligned}$$

Thus, the non-trivial solutions of (P_1') for the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{N}$ are the eigenfunctions given by:

$$G_n(\theta)=C_n\cos\big(n\theta\big)+D_n\sin\big(n\theta\big),\quad (n=0,1,2,\dots),$$

where C_n and D_n are arbitrary constants.

Resolution of the problem (P_2').

Substituting $\lambda_n = n^2$, $\quad n \in \mathbb{N}$ into (P_2'), we get the equation

$$r^2F_n''(r)+rF_n'(r)-n^2F_n(r)=0,\quad n\in\mathbb{N}. \tag{1.24}$$

1) If $\lambda_n = n = 0$ in equation (1.24), we get

$$\begin{aligned} rF_0''(r)+F_0'(r)=0 &\Longleftrightarrow \Big(rF_0'(r)\Big)'=0\\ &\Longleftrightarrow rF_0'(r)=B_0\\ &\Longleftrightarrow F_0(r)=A_0+B_0\ln|r|, \end{aligned}$$

where A_0 and B_0 are arbitrary constants.

2) If $\lambda_n = n$, with $n \in \mathbb{N}^*$, the equation (1.24) is an Euler equation. Let us assume $F_n(r) = r^\alpha$

in equation (1.24), and we obtain

$$\alpha(\alpha-1)r^{\alpha}+\alpha r^{\alpha}-n^{2}r^{\alpha}=0 \Longleftrightarrow \alpha(\alpha-1)+\alpha-n^{2}=0,$$

so that $\alpha=\pm n$, and the solution is of the form: $F_n(r)=A_n r^n+B_n r^{-n}$, for $n\geq 1$, where A_n and B_n are arbitrary constants.

Finally, the general solution of the problem (P') is given by, for $r>0$:

$$w_n(r,\,\theta)=A_0+B_0\ln(r)+\Big(A_n r^n+B_n r^{-n}\Big)\Big(C_n\cos\big(n\theta\big)+D_n\sin\big(n\theta\big)\Big),\quad n\in\mathbb{N}.$$

However, according to the problem (P_1'), we must set $B_n=0$, for $n\in\mathbb{N}$, to ensure that the product $F(r)G(\theta)$ defines a continuous function at the origin of $\mathbb{R}^2$. Thus, the final general solution of the problem (P') by the method of separation of variables is written as:

$$w_n(r,\,\theta)=r^n\Big(C_n'\cos\big(n\theta\big)+D_n'\sin\big(n\theta\big)\Big),\quad n\in\mathbb{N},$$

where C_n' and D_n' are arbitrary constants with $C_n'=A_nC_n$ and $D_n'=A_nD_n$. Also, applying the Fourier method, we seek the solution u in the form:

$$w(r,\,\theta)=\frac{C_0'}{2}+\sum_{n=1}^{\infty}r^n\Big(C_n'\cos\big(n\theta\big)+D_n'\sin\big(n\theta\big)\Big),\quad n\in\mathbb{N}, \tag{1.25}$$

which is a solution of the problem (P') that defines a continuous function w at the origin of $\mathbb{R}^2$. The next step is to choose the coefficients C_n' and D_n', for $n=0,1,2,\ldots$, such that w satisfies the condition $w\big(r=a,\,\theta\big)=h(\theta)$, for $0\leq\theta\leq 2\pi$, i.e.:

$$\begin{aligned} w\big(r=a,\,\theta\big)=h(\theta)&=\frac{C_0'}{2}+\sum_{n=1}^{\infty}a^n\Big(C_n'\cos\big(n\theta\big)+D_n'\sin\big(n\theta\big)\Big)\\ &=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\Big(\alpha_n\cos\big(n\theta\big)+\beta_n\sin\big(n\theta\big)\Big),\quad n\in\mathbb{N},\end{aligned}$$

where $\alpha_n=a^nC_n'$ and $\beta_n=a^nD_n'$. The coefficients α_n and β_n are thus proportional to the trigonometric Fourier coefficients of the 2π-periodic function that coincides with h on $[0,2\pi]$. In this case, they are determined by:

$$\begin{cases} \alpha_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} h(\theta)\cos(n\theta)\,d\theta, \quad (n = 0, 1, 2, \dots), \\ \beta_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} h(\theta)\sin(n\theta)\,d\theta, \quad (n = 1, 2, 3, \dots). \end{cases} \tag{1.26}$$

Consequently, by substituting (1.26) into (1.25), we obtain

$$w(r,\,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \Big(\alpha_n\cos\big(n\theta\big) + \beta_n\sin\big(n\theta\big)\Big),$$

which is a solution to the problem (P').

1.6 Poisson kernel

There is another way to solve the Dirichlet problem—with the help of an integral kernel. That is, we will find a function $P(r, \theta, \alpha)$ called the *Poisson kernel*[3] such that

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta, \alpha)\, g(\alpha)\, d\alpha.$$

While the integral will generally not be solvable analytically, it can be evaluated numerically. In fact, unless the boundary data is given as a Fourier series already, it may be much easier to numerically evaluate this formula as there is only one integral to evaluate.

---

[3] Named for the French mathematician Siméon Denis Poisson (1781–1840).

We must use a different dummy variable for the integration and hence we use α instead of θ.

$$\begin{aligned}
u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\
&= \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\alpha)\,d\alpha\right)}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(\alpha)\cos(n\alpha)\,d\alpha\right)}_{a_n} r^n \cos(n\theta) + \\
&\qquad + \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} g(\alpha)\sin(n\alpha)\,d\alpha\right)}_{b_n} r^n \sin(n\theta) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(g(\alpha) + 2\sum_{n=1}^{\infty} g(\alpha)\cos(n\alpha)\,r^n\cos(n\theta) + g(\alpha)\sin(n\alpha)\,r^n\sin(n\theta)\right) d\alpha \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left(1 + 2\sum_{n=1}^{\infty} r^n\big(\cos(n\alpha)\cos(n\theta) + \sin(n\alpha)\sin(n\theta)\big)\right)}_{P(r,\theta,\alpha)} g(\alpha)\,d\alpha.
\end{aligned}$$

OK, so we have what we wanted, the expression in the parentheses is the Poisson kernel, $P(r,\theta,\alpha)$. However, we can do a lot better. It is still given as a series, and we would really like to have a nice simple expression for it. We must work a little harder. The trick is to rewrite everything in terms of complex exponentials. Let us work just on the kernel.

$$\begin{aligned}
P(r,\theta,\alpha) &= 1 + 2\sum_{n=1}^{\infty} r^n\big(\cos(n\alpha)\cos(n\theta) + \sin(n\alpha)\sin(n\theta)\big) \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \cos\big(n(\theta-\alpha)\big) \\
&= 1 + \sum_{n=1}^{\infty} r^n\big(e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}\big) \\
&= 1 + \sum_{n=1}^{\infty}\big(re^{i(\theta-\alpha)}\big)^n + \sum_{n=1}^{\infty}\big(re^{-i(\theta-\alpha)}\big)^n.
\end{aligned}$$

In the expression above, we recognize the *geometric series.* Recall from calculus that if z is a complex number where $|z| < 1$, then

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Note that n starts at 1, and that is why we have the z in the numerator. It is the standard geometric series multiplied by z. We can use $z = re^{i(\theta-\alpha)}$, as lo and behold $|re^{i(\theta-\alpha)}| = r < 1$. We continue with the computation.

$$\begin{aligned}
P(r,\theta,\alpha) &= 1+\sum_{n=1}^{\infty}\left(re^{i(\theta-\alpha)}\right)^n+\sum_{n=1}^{\infty}\left(re^{-i(\theta-\alpha)}\right)^n \\
&= 1+\frac{re^{i(\theta-\alpha)}}{1-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{1-re^{-i(\theta-\alpha)}} \\
&= \frac{\left(1-re^{i(\theta-\alpha)}\right)\left(1-re^{-i(\theta-\alpha)}\right)+\left(1-re^{-i(\theta-\alpha)}\right)re^{i(\theta-\alpha)}+\left(1-re^{i(\theta-\alpha)}\right)re^{-i(\theta-\alpha)}}{\left(1-re^{i(\theta-\alpha)}\right)\left(1-re^{-i(\theta-\alpha)}\right)} \\
&= \frac{1-r^2}{1-re^{i(\theta-\alpha)}-re^{-i(\theta-\alpha)}+r^2} \\
&= \frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}.
\end{aligned}$$

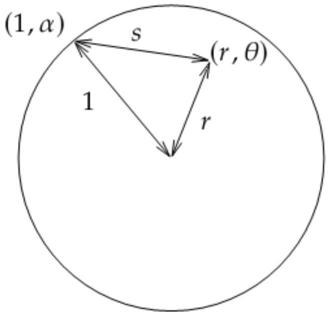
That is a formula we can live with. The solution to the Dirichlet problem using the Poisson kernel is

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}g(\alpha)\,d\alpha.$$

Sometimes the formula for the Poisson kernel is given together with the constant $\frac{1}{2\pi}$, in which case we should, of course, not leave it in front of the integral. Sometimes the limits of the integral are given as 0 to 2π; everything inside is 2π-periodic in α, so this does not change the integral.

Let us not leave the Poisson kernel without explaining its geometric meaning. Let s be the distance from (r,θ) to $(1,\alpha)$. This distance s in polar coordinates is given precisely by the square root of $1-2r\cos(\theta-\alpha)+r^2$. That is, the Poisson kernel is really the formula

$$\frac{1-r^2}{s^2}.$$



One final note we make about the formula is that it is really a weighted average of the boundary values. First, we look at what happens at the origin, that is, when $r = 0$:

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-0^2}{1-2(0)\cos(0-\alpha)+0^2} g(\alpha)\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\alpha)\, d\alpha. \end{aligned}$$

So $u(0,0)$ is precisely the average value of $g(\theta)$ and therefore the average value of u on the boundary. This is a general feature of harmonic functions, the value at some point p is equal to the average of the values on a circle centered at p.

What the formula says at other points inside the circle is that the value of the solution is a weighted average of the boundary data $g(\theta)$. The kernel is bigger when $(1,\alpha)$ is closer to (r,θ). Therefore, when computing $u(r,\theta)$, we give more weight to the values $g(\alpha)$ when $(1,\alpha)$ is closer to (r,θ) and less weight to the values $g(\alpha)$ when $(1,\alpha)$ far from (r,θ).

1.7 Laplace's equation in three dimension

In this section, we focus on solving Laplace's equation in dimension $n = 3$ using the method of separation of variables. The mathematical model is written as follows:

$$\Delta u(x,y,z) = u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0, \quad \text{in} \quad D \tag{1.27}$$

where $D = \left\{0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\right\}$ with the boundary conditions:

$$\begin{cases} u(0,y,z) = 0, \quad u(\pi,y,z) = 0, \quad u(x,0,z) = 0 \\ u(x,\pi,z) = 0, \quad u(x,y,\pi) = 0, \quad u(x,y,0) = g(x,y) \end{cases} \tag{1.28}$$

where g is a given function.

According to the method of separation of variables, we assume that any non-trivial solution has the form:

$$u(x,y,z) = X(x)Y(y)Z(z) \tag{1.29}$$

which satisfies the three homogeneous conditions in (1.28). We have omitted the non-homogeneous condition $u(\pi,y,z) = g(y,z)$.

By substituting this solution into the PDE (1.27) and separating variables, we obtain:

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0 \tag{1.30}$$

where X'' is the second derivative of X with respect to x, Y'' is the second derivative of Y with respect to y, and Z'' is the second derivative of Z with respect to z.

Dividing equation (1.30) by $X(x)Y(y)Z(z)$ gives:

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}=0 \Longrightarrow \frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\frac{Z''(z)}{Z(z)}. \tag{1.31}$$

In (1.31), the left-hand term is independent of z, whereas the right-hand term is a function of z only. Consequently, both expressions must be constant, and we can write:

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\frac{Z''(z)}{Z(z)}=\lambda_1, \quad \lambda_1\in\mathbb{R}. \tag{1.32}$$

Similarly, by following the same reasoning, we obtain:

$$\frac{X''(x)}{X(x)}=\lambda_1-\frac{Y''(y)}{Y(y)}=\lambda_2, \quad \lambda_2\in\mathbb{R}. \tag{1.33}$$

Also, according to the boundary conditions (1.28), the solution u satisfies the three homogeneous conditions. Thus, we obtain:

$$\begin{cases} u(0,y,z)=0 \Longrightarrow X(0)Y(y)Z(z)=0, \quad x\in[0,\pi] \Longrightarrow X(0)=0, \\ u(\pi,y,z)=0 \Longrightarrow X(\pi)Y(y)Z(z)=0, \quad x\in[0,\pi] \Longrightarrow Z(\pi)=0, \\ u(x,0,z)=0 \Longrightarrow X(0)Y(0)Z(z)=0, \quad y\in[0,\pi] \Longrightarrow Y(0)=0, \\ u(x,\pi,z)=0 \Longrightarrow X(x)Y(\pi)Z(z)=0, \quad y\in[0,\pi] \Longrightarrow Y(\pi)=0, \\ u(x,y,\pi)=0 \Longrightarrow X(x)Y(y)Z(\pi)=0, \quad z\in[0,\pi] \Longrightarrow Z(\pi)=0. \end{cases} \tag{1.34}$$

In summary, according to (1.32), (1.33), and (1.34), we need to study the following problems:

$$(p_1)\begin{cases} X''(x)-\lambda_2 X(x)=0,\ 0<x<\pi, \\ X(0)=0,\ \ X(\pi)=0, \end{cases}$$

$$(p_2)\begin{cases} Y''(y)-\big(\lambda_1-\lambda_2\big)Y(y)=0,\ 0<y<\pi, \\ Y(0)=0,\ \ Y(\pi)=0 \end{cases}$$

and

$$(p_3)\begin{cases} Z''(z)+\lambda_1 Z(z)=0, \quad 0<z<\pi, \\ Z(\pi)=0. \end{cases}$$

Solution of problem (p_1)

We distinguish three possible cases for λ_2. **Case 1:** If $\lambda_2 = k^2 > 0$, with k being a nonzero real number, then the problem (p_1) admits a general solution of the form:

$$X(x) = A_1 e^{-kx} + A_2 e^{kx}$$

where A_1 and A_2 are positive constants. According to the conditions $X(0) = 0,\ X(\pi) = 0$, we obtain:

$$(S_1)\ \begin{cases} A_1 + A_2 = 0 \\ A_1 e^{-k\pi} + A_2 e^{k\pi} = 0. \end{cases}$$

The system (S_1) has only one solution, which is the trivial solution $(A_1, A_2) = (0,0)$. Indeed, the determinant of the system (S_1) is:

$$\begin{vmatrix} 1 & 1 \\ e^{-k\pi} & e^{k\pi} \end{vmatrix} = e^{k\pi} - e^{-k\pi} = \frac{e^{2k\pi} - 1}{e^{k\pi}} \neq 0.$$

Otherwise,

$$e^{2k\pi} - 1 = 0 \Longrightarrow e^{2k\pi} = 1 \Longrightarrow 2k\pi = 0.$$

But this is absurd because μ is different from zero. Thus, we can conclude that:

$$\begin{pmatrix} 1 & 1 \\ e^{-k\pi} & e^{k\pi} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since the matrix $\begin{pmatrix} 1 & 1 \\ e^{-k\pi} & e^{k\pi} \end{pmatrix}$ is invertible. Thus, in the case where $\lambda_2 = k^2 > 0$, we have $X(x) \equiv 0$ and consequently $u(x, y, z) \equiv 0$ for all $0 < x < \pi$. Therefore, we must exclude the case $\lambda_2 > 0$. **Case 2:** If $\lambda_2 = 0$, we have $X(x) = A_1 x + A_2$, where A_1 and A_2 are arbitrary real numbers. However, from the boundary conditions $X(\pi) = 0$ and $X(0) = 0$, we obtain

$$\begin{cases} A_2 = 0, \\ A_1 \pi + A_2 = 0 \end{cases} \Longrightarrow \begin{cases} A_2 = 0, \\ A_1 = 0. \end{cases}$$

Thus, in the case where $\lambda_2 = 0$, we obtain $X(x) \equiv 0$, leading to $u(x, y, z) \equiv 0$ for all $0 < x < \pi$. Consequently, we must also exclude the case $\lambda_2 = 0$.

Case 3: If $\lambda_2 = -k^2 < 0$, the general solution to problem (p_1) is given by

$$X(x) = A_1 \sin\big(kx\big) + A_2 \cos\big(kx\big),$$

where A_1 and A_2 are arbitrary real numbers. From the boundary condition $X(0) = 0$, we obtain $A_2 = 0$. Hence, a non-trivial solution can be written as

$$X(x) = A_1 \sin\big(kx\big),$$

with $A_1 \neq 0$.

Moreover, since $X(\pi) = 0$, we must have $\sin\big(k\pi\big) = 0$, which implies $k\pi = n\pi$, with $n \in \mathbb{Z}$. This leads to $\lambda_2 = -n^2$. The values $\lambda_2 = -n^2$ are called eigenvalues, and the functions $X(x) = A_1^n \sin(nx)$ are the characteristic functions of problem (p_1), where A_1^n are constants. Since $\sin(-x) = -\sin(x)$ for all $x \in \mathbb{R}$, it is sufficient to consider the eigenvalues λ_2 and the characteristic functions X for all $n \in \mathbb{N}^*$.

Solution of problem (p_2)

Let $\lambda = \lambda_1 - \lambda_2$. As before, we distinguish three cases for λ.

Case 1: If $\lambda = \mu^2 > 0$, where μ is a nonzero real number, then problem (p_2) admits a general solution of the form:

$$Y(y) = B_1 e^{-\mu y} + B_2 e^{\mu y},$$

where B_1 and B_2 are constants. From the conditions $Y(0) = 0$ and $Y(\pi) = 0$, we obtain the system:

$$(S_2) \begin{cases} B_1 + B_2 = 0, \\ B_1 e^{-\mu\pi} + B_2 e^{\mu\pi} = 0. \end{cases}$$

The system (S_2) has only one solution, which is the trivial solution $(B_1, B_2) = (0, 0)$. Indeed, the determinant of the system (S_2) is given by:

$$\begin{vmatrix} 1 & 1 \\ e^{-\mu\pi} & e^{\mu\pi} \end{vmatrix} = e^{\mu\pi} - e^{-\mu\pi} = \frac{e^{2\mu\pi} - 1}{e^{\mu\pi}} \neq 0.$$

Otherwise, we would have:

$$e^{2\mu\pi} - 1 = 0 \Longrightarrow e^{2\mu\pi} = 1 \Longrightarrow 2\pi\mu = 0.$$

However, this is absurd because μ is nonzero. Thus, we conclude that:

$$\begin{pmatrix} 1 & 1 \\ e^{-\mu\pi} & e^{\mu\pi} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since the matrix

$$\begin{pmatrix} 1 & 1 \\ e^{-\mu\pi} & e^{\mu\pi} \end{pmatrix}$$

is invertible. Thus, in the case where $\lambda = \mu^2 > 0$, we obtain $Y(y) \equiv 0$, which implies that $u(x, y, z) \equiv 0$ for all $0 < y < \pi$. Therefore, we must exclude the case $\lambda > 0$. **Case 2:** If $\lambda = 0$, we have $Y(y) = B_1 y + B_2$, where B_1 and B_2 are arbitrary real numbers. However, from the conditions $Y(\pi) = 0$ and $Y(0) = 0$, we obtain:

$$\begin{cases} B_2 = 0, \\ B_1\pi + B_2 = 0 \end{cases} \Longrightarrow \begin{cases} B_2 = 0, \\ B_1 = 0. \end{cases}$$

Thus, in the case where $\lambda = 0$, we get $Y(y) \equiv 0$ and $u(x, y, z) \equiv 0$ for all $0 < y < \pi$. Therefore, we must also exclude the case $\lambda = 0$.

Case 3: If $\lambda_2 = -k^2 < 0$, the general solution of problem (p_1) is given by:

$$Y(y) = B_1 \sin\big(\mu y\big) + B_2 \cos\big(\mu y\big),$$

where B_1 and B_2 are arbitrary real numbers. From the condition $Y(0) = 0$, we obtain $B_2 = 0$. Hence, a non-trivial solution is given by $Y(y) = B_1 \sin\big(\mu y\big)$ with $B_1 \neq 0$.

Additionally, since $Y(\pi) = 0$, we have $\sin\big(\mu\pi\big) = 0$, which implies $\mu\pi = m\pi$ for some $m \in \mathbb{Z}$, leading to $\lambda = -m^2$. The values $\lambda = -m^2$ are called eigenvalues, and the functions $Y(y) = B_1^n \sin(my)$ are the characteristic functions of problem (p_2), where B_1^n are constants. Since $\sin(-y) = -\sin(y)$ for all $y \in \mathbb{R}$, it suffices to consider the eigenvalues λ and the characteristic functions Y for all $m \in \mathbb{N}^*$.

Now, if we consider problem (p_3) for eigenvalues $\lambda_1 - \lambda_2 = -m^2$ with $\lambda_2 = -n^2$, for all n, $m \in \mathbb{N}^*$, we obtain $\lambda_1 = -m^2 - n^2$, and the general solution of problem (p_3) is given by:

$$Z(z) = C_1 e^{z\sqrt{n^2+m^2}} + C_2 e^{-z\sqrt{n^2+m^2}},$$

where C_1 and C_2 are arbitrary constants.

Moreover, we have:

$$Z(\pi) = 0 \Longrightarrow C_1 e^{\pi\sqrt{n^2+m^2}} + C_2 e^{-\pi\sqrt{n^2+m^2}} = 0 \Longrightarrow C_2 = -C_1 e^{2\pi\sqrt{n^2+m^2}}.$$

Substituting this into the solution for Z, we obtain:

$$\begin{aligned} Z(z) =& C_1 e^{z\sqrt{n^2+m^2}} - C_1 e^{2\pi\sqrt{n^2+m^2}} e^{-z\sqrt{n^2+m^2}} \\ =& C_1 e^{\pi\sqrt{n^2+m^2}} \left[e^{(z-\pi)\sqrt{n^2+m^2}} - e^{-(z-\pi)\sqrt{n^2+m^2}} \right] \\ =& 2C_1 e^{\pi\sqrt{n^2+m^2}} \sinh\Big(\sqrt{n^2+m^2}(z-\pi)\Big) \\ =& C_1^{nm} \sinh\Big(\sqrt{n^2+m^2}(z-\pi)\Big), \quad \text{where} \quad C_1^{nm} = 2C_1 e^{\pi\sqrt{n^2+m^2}}. \end{aligned}$$

Since

$$\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \text{for all} \quad \theta \in \mathbb{R}.$$

Finally, combining all the results, we obtain a solution of the form:

$$u(x,y,z) = X(x)Y(y)Z(z) = C_{nm} \sin(nx)\sin(my)\sinh\Big(\sqrt{n^2+m^2}(z-\pi)\Big), \tag{1.35}$$

which is a general solution to the PDE (1.27) for $n,\ m \in \mathbb{N}^*$, where $C_{nm} = A_1^n B_1^m C_1^{nm}$ are constants.

This problem is linear and homogeneous. In this case, the principle of superposition applies, and:

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin(nx)\sin(my)\sinh\Big(\sqrt{n^2+m^2}(z-\pi)\Big) \tag{1.36}$$

is also a solution of the problem (1.27)–(1.28), where C_{nm} are constants to be determined.

Setting $z = 0$ in (1.36) and using the condition $u(x,y,0) = g(x,y)$, we obtain:

$$\begin{aligned} g(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sinh\Big(\sqrt{n^2+m^2}(\pi)\Big)\sin(nx)\sin(my) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[C_{nm} \sinh\Big(\sqrt{n^2+m^2}(\pi)\Big)\right]\sin(nx)\sin(my) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm} \sin(nx)\sin(my), \quad x \in [0,\pi],\ y \in [0,\pi], \end{aligned} \tag{1.37}$$

where:

$$\alpha_{nm} = C_{nm} \sinh\Big(\sqrt{n^2+m^2}(\pi)\Big), \quad (n = 1,2,...),\ (m = 1,2,...). \tag{1.38}$$

Multiplying equation (1.37) by $\sin(px)$ and integrating over $[0,\pi]$, we obtain:

$$\int_0^\pi g(x,y)\sin(px)dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{nm}\left[\int_0^\pi \sin(nx)\sin(px)dx\right]\sin(my).$$

But we have

$$\int_0^\pi \sin(nx)\sin(px)dx = \begin{cases}\pi/2, & \text{if} \quad n=p\\ 0, & \text{if} \quad n\neq p\end{cases}$$

thus,

$$\int_0^\pi g(x,y)\sin(px)dx = \sum_{m=1}^{\infty}\alpha_{pm}\frac{\pi}{2}\sin(my).$$

Also, by multiplying this equality by $\sin(qx)$ and integrating over $[0,\pi]$, we obtain

$$\int_0^\pi\int_0^\pi g(x,y)\sin(px)\sin(qy)dxdy = \sum_{m=1}^{\infty}\alpha_{pm}\frac{\pi}{2}\int_0^\pi \sin(my)\sin(qy)dy.$$

Now,

$$\int_0^\pi \sin(my)\sin(qy)dy = \begin{cases}\pi/2, & \text{if} \quad m=q\\ 0, & \text{if} \quad m\neq q.\end{cases}$$

Finally, we obtain:

$$\int_0^\pi\int_0^\pi g(x,y)\sin(px)\sin(qy)dxdy = \alpha_{pq}\frac{\pi^2}{4}.$$

From which we deduce

$$C_{nm}\sinh\left(\sqrt{n^2+m^2}(\pi)\right) = \alpha_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(x,y)\sin(nx)\sin(my)dxdy,$$

where n, $m \in \mathbb{N}^*$. Finally, we have

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{nm}\frac{\sinh\left(\sqrt{n^2+m^2}(z-\pi)\right)}{\sinh\left(\sqrt{n^2+m^2}(\pi)\right)}\sin(nx)\sin(my) \tag{1.39}$$

which is a solution of the PDE (1.27) with the boundary conditions (1.28).

1.8 Fundamental solution and Green functions

1.8.1 Green's formulas

We first give a theorem without proof, which will be used as a basis for the proof of Green's formulas.

Theorem 2. ***(Ostrogradski's Theorem)****. Let $\overrightarrow{V} = (V_1, ..., V_n)$ be a vector-valued function of class C^1 in a bounded open set $\Omega \subset \mathbb{R}^n$ and continuous on $\overline{\Omega}$. Then:*

$$\int\limits_{\Omega} div \overrightarrow{V} \, dx = \int\limits_{\partial\Omega} \left(\overrightarrow{V}, \overrightarrow{\boldsymbol{n}} \right)_{\mathbb{R}^n} d\sigma, \tag{1.40}$$

where $\overrightarrow{\boldsymbol{n}}$ is the unit normal vector at each point on $\partial\Omega$, oriented outward from Ω, and $d\sigma$ represents the surface element.

The First Green's Formula. Let $\overrightarrow{V} = \text{grad } \varphi$. Suppose that φ is of class $C^2(\overline{\Omega})$. From (1.40) and since $\text{div}(\text{grad } \varphi) = \Delta\varphi$, we obtain:

$$\int\limits_{\Omega} \Delta\varphi \, dx = \int\limits_{\partial\Omega} \frac{\partial\varphi}{\partial\mathbf{n}} \, d\sigma. \tag{1.41}$$

The Second Green's Formula. Let φ be of class $C^1(\overline{\Omega})$ and ψ of class $C^2(\overline{\Omega})$. Applying Ostrogradski's theorem for $\overrightarrow{V} = \varphi \text{ grad } \psi$, we obtain:

$$\int\limits_{\Omega} \varphi\Delta\psi \, dx + \int\limits_{\Omega} (\text{grad}\, \varphi, \, \text{grad}\, \psi)_{\mathbb{R}^n} \, dx = \int\limits_{\partial\Omega} \varphi \frac{\partial\psi}{\partial\mathbf{n}} \, d\sigma, \tag{1.42}$$

because $\text{div}\overrightarrow{V} = (\text{grad}\, \varphi, \, \text{grad}\, \psi)_{\mathbb{R}^n} + \varphi\Delta\psi$.

The Third Green's Formula. If φ and ψ are of class $C^2(\overline{\Omega})$, then, from (1.40), we have:

$$\int\limits_{\Omega} (\varphi\Delta\psi - \psi\Delta\varphi) \, dx = \int\limits_{\partial\Omega} \left(\varphi\frac{\partial\psi}{\partial\mathbf{n}} - \psi\frac{\partial\varphi}{\partial\mathbf{n}} \right) d\sigma. \tag{1.43}$$

Remark 3. *From Green's formulas, we obtain the following properties of harmonic functions:*

(1) Let u be a harmonic function on Ω. If we take $\varphi = \psi = u$ in the second Green's formula, we get:

$$\int\limits_{\Omega} \left| grad\, u \right|^2 \, dx = \int\limits_{\partial\Omega} u \frac{\partial u}{\partial \boldsymbol{n}} \, d\sigma,$$

since $\Delta u = 0$. Therefore, harmonic functions satisfy the following inequality:

$$\int\limits_{\partial\Omega} u \frac{\partial u}{\partial \boldsymbol{n}}\, d\sigma \geq 0.$$

(2) If $\varphi = u$ and $\psi = 1$ in the third Green's formula, then:

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}}\, d\sigma = 0$$

for all harmonic functions u. This property is called ***Gauss' Theorem*** *for harmonic functions.*

1.8.2 Green functions

Let (ξ, η) and (x, y) denote an arbitrary fixed point and a variable point, respectively, in the xy-plane. We recall that in polar coordinates with pole at (ξ, η), Laplace equation takes the form

$$\Delta u(r,\theta) = u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, \tag{1.44}$$

where

$$r = \sqrt{\left(x-\xi\right)^2 + \left(y-\eta\right)^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{(y-\eta)}{(x-\xi)}.$$

If u is a solution of (1.44) that depends only on r, then u satisfies the ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0,$$

thus, u must be of the form

$$u(r,\, \theta) = c_1 \ln(r) + c_2, \quad r > 0$$

where C_1 and C_2 are arbitrary constants. If we choose $C_1 = \frac{1}{2\pi}$ and $C_2 = 0$, then the particular solution

$$U(r,\, \theta) = \frac{1}{2\pi}\ln(r) \tag{1.45}$$

is called the fundamental solution of the two-dimensional Laplace equation.

Remark 4. *The fundamental solution has the property of being harmonic throughout the xy-plane except at the pole (ξ, η), where it becomes logarithmically infinite. It is the simplest conceivable* *solution having this type of singularity at a point. The essential role played by the fundamental solution (1.45) is that it leads to an integral formula that expresses the value of any harmonic function u inside a domain in terms of the values of u and its normal derivative* $\frac{\partial u}{\partial n}$ *on the boundary of the domain. This makes possible for us to obtain explicit representations for the solutions of the Dirichlet problem and the Neumann problem in terms of kernel function as Green function and Neumann function, respectively.*

Our basic tool is the Green's second identity

$$\iint\limits_D \Big(u\Delta v - v\Delta u\Big) dxdy = \int\limits_C \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right) ds \tag{1.46}$$

over a domain D with a smooth boundary C, for all pour functions $u \in C^2(\overline{D})$ and $v \in C^2(\overline{D})$.

We shall take v in (1.45) to be the fundamental solution U with the pole (ξ, η) in D. Since U is singular at the pole, we delete from D a small disk about (ξ, η) with radius ε and boundary C_0

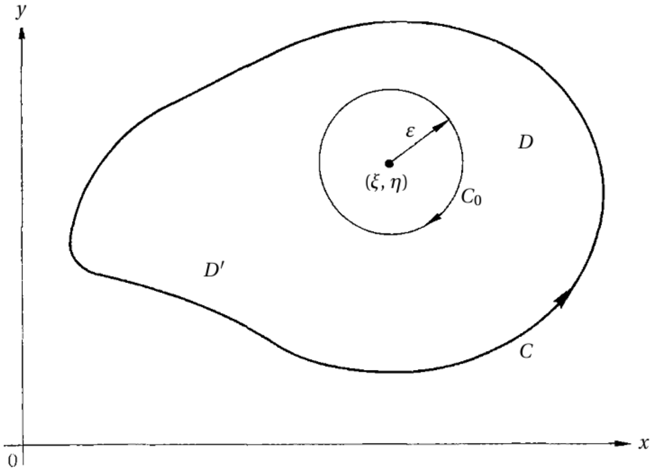


Figure 1.3: Fundamental solution.

Applying (1.46) to the punctured domain D' bounded by C and C_0, and noting that $\Delta U = 0$ throughout D', we find

$$-\iint\limits_D \ln(r)\Delta u dxdy = \int\limits_{C+C_0} \left(u\frac{\partial \ln(r)}{\partial \mathbf{n}} - \ln(r)\frac{\partial u}{\partial \mathbf{n}}\right) ds \tag{1.47}$$

where we have dropped the common factor $\frac{1}{2\pi}$. This relation holds for all values of ε no matter how small. We shall show that the integral along C_0 yields the value $-2\pi u(\xi, \eta)$ when ε is allowed to approach zero.

On C_0 we note that $r = \varepsilon = const$ and the outward normal vector is opposite in direction to the radius vector. Thus,

$$\left.\frac{\partial \ln(r)}{\partial \mathbf{n}}\right|_{r=\varepsilon} = -\frac{d}{dr}\ln(r)\bigg|_{r=\varepsilon} = -\frac{1}{\varepsilon}.$$

So, by introducing polar coordinates, we see that

$$\lim_{\varepsilon \longrightarrow 0}\int\limits_{C_0} u\frac{\partial \ln(r)}{\partial \mathbf{n}}ds = \lim_{\varepsilon \longrightarrow 0}\int\limits_0^{2\pi} u\big(\xi + \varepsilon\cos\theta,\, \eta + \varepsilon\sin\theta\big)\left(-\frac{1}{\varepsilon}\right)\varepsilon d\theta = -2\pi u(\xi,\, \eta). \tag{1.48}$$

Since the first derivative of u are continuous in D, there is a constant M such that $\frac{\partial u}{\partial n} \leq M$ on C_0. Therefore,

$$\left|\int\limits_{C_0} u\frac{\partial \ln(r)}{\partial \mathbf{n}}ds\right| \leq M\int\limits_0^{2\pi}\big|\ln(\varepsilon)\big|\varepsilon d\theta \leq 2\pi M\big|\ln(\varepsilon)\big| \longrightarrow 0, \quad \text{when} \quad \varepsilon \longrightarrow 0. \tag{1.49}$$

which tends to zero with ε. Thus letting ε approach zero in (1.47) and using (1.48) and (1.49), we obtain the integral formula

$$u(\xi,\, \eta) = \frac{1}{2\pi}\int\limits_C\left(u\frac{\partial \ln(r)}{\partial \mathbf{n}} - \ln(r)\frac{\partial u}{\partial \mathbf{n}}\right)ds + \frac{1}{2\pi}\iint\limits_D \ln(r)\Delta u dxdy. \tag{1.50}$$

This expresses the value of u at any point in D terms of the values of u and $\frac{\partial u}{\partial n}$ on the boundary C and of Δu in D. In particular when $u = 1$ we obtain an important property of the fundamental solution, namely

$$\frac{1}{2\pi}\int\limits_C \frac{\partial \ln(r)}{\partial \mathbf{n}} = 1.$$

Formula (1.50) does not immediately yield a representation for the solution of the *Dirichlet problem* or the *Neumann problem*, since it involves the values of both u and its normal derivative $\frac{\partial u}{\partial n}$ on C. We shall therefore convert it into an integral formula from which it will be possible to eliminate u or $\frac{\partial u}{\partial n}$.

Let $g = g(x, y, \xi, \eta)$ be a function of (x, y) that depends on the pole (ξ, η) such that g is harmonic throughout the domain D. Applying the Green formula for the function u and g, we then have

$$\iint\limits_D g\Delta u dxdy + \int\limits_C\left(u\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial u}{\partial \mathbf{n}}\right)ds = 0. \tag{1.51}$$

Combining this with (1.50) with (1.51), we thus obtain

$$u(\xi,\,\eta)=\int\limits_{C}\left(u\frac{\partial G}{\partial \mathbf{n}}-G\frac{\partial u}{\partial \mathbf{n}}\right)ds+\iint\limits_{D}G\Delta u dxdy \tag{1.52}$$

where $G=G(x,y,\xi,\eta)$ with

$$G\big(x,y\,;\,\xi,\eta\big)=\frac{1}{2\pi}\ln(r)+g\big(x,y\,;\,\xi,\eta\big). \tag{1.53}$$

Now, if the harmonic function g can be determined in such a way that G or $\frac{\partial G}{\partial n}$ vanishes on C , the term involving $\frac{\partial u}{\partial n}$ or u in (1.52) disappears, and the resulting formula yields a representation for the solution of the corresponding Dirichlet or Neumann problem.

We focus our attention here on the problem of determining the function g such that G vanishes on the boundary C of the domain D. It is clear that if g is a solution of the boundary value problem

$$\begin{cases}\Delta g=\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial y^2}=0, & \text{in} \quad D,\\ g(x,y;\xi,\eta)=-\dfrac{1}{2\pi}\ln(r), & \text{on} \quad C\end{cases} \tag{1.54}$$

for each point (ξ,η) in D, then the function G defined by (1.53) vanishes when (x,y) lies on C. Thus , we obtain from (1.52) the specific representation formula

$$u(\xi,\,\eta)=\int\limits_{C}u(x,y)\frac{\partial G}{\partial \mathbf{n}}\big(x,y\,;\,\xi,\eta\big)ds+\iint\limits_{D}G\big(x,y\,;\,\xi,\eta\big)\Delta u(x,y)dxdy$$

for the solution u of the Dirichlet problem

$$\begin{cases}\Delta u=q(x,y), & \text{dans} \quad D,\\ u=f(x,y), & \text{sur} \quad C\end{cases}$$

in terms of the function (1.53) and its normal derivative on the boundary.

Theorem 3. *Let D be a bounded domain with a smooth boundary ∂D, $q\in C(\overline{D})$ and $f\in C(\partial D)$. Let $u\in C^2(\overline{D})$ be a solution of the following problem*

$$\begin{cases}\Delta u=q(x,y), & \textit{in} \quad D,\\ u=f(x,y), & \textit{on} \quad \partial D.\end{cases} \tag{1.55}$$

Then

$$u(\xi,\,\eta) = \int\limits_{\partial D} f(x,y)\frac{\partial G}{\partial \mathbf{n}}\big(x,y\,;\,\xi,\eta\big)ds + \iint\limits_{D} G\big(x,y\,;\,\xi,\eta\big)q(x,y)dxdy \tag{1.56}$$

*where G is **Green function** defined in (1.53) and (1.54).*

1.8.3 Proprieties of Green functions

1) For each $(\xi,\,\eta)\in D$, G satisfies the Laplace's equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0, \qquad \forall (x,y)\in D$$

except at (ξ,η).

2) $G\big(x,y\,;\,\xi,\eta\big) = 0$ for each $(x,y)\in\partial D$.

3) G has logarithmic singularity at (ξ,η) such that

$$\int\limits_{\partial D}\frac{\partial G}{\partial \mathbf{n}}ds \equiv 1.$$

4) G is symmetric with respect to the points (x,y) and $(\xi,\,\eta)$, namely

$$G\big(x,y\,;\,\xi,\eta\big) = G\big(\xi,\eta\,;\,x,y\big).$$

1.8.4 Examples of Green's functions

In this section we shall present examples of Green's functions for the Laplace's equation for special domains in the xy-plane.

Example 3. *Let $\mathbb{R}^2_+ = \big\{(x,y)|\quad y>0\big\}$ the half-plane $y>0$. Find the Green function of the following Laplace equation $\Delta u(x,y)=0$ in $\mathbb{R}^2_+$.*

Solution Let $P : (\xi, \eta)$ denote the pole of the Green's function in the half-plane $y > 0$ and let $Q : (x, y)$ $y > 0$, be any point whose distance from the pole is denote by r.

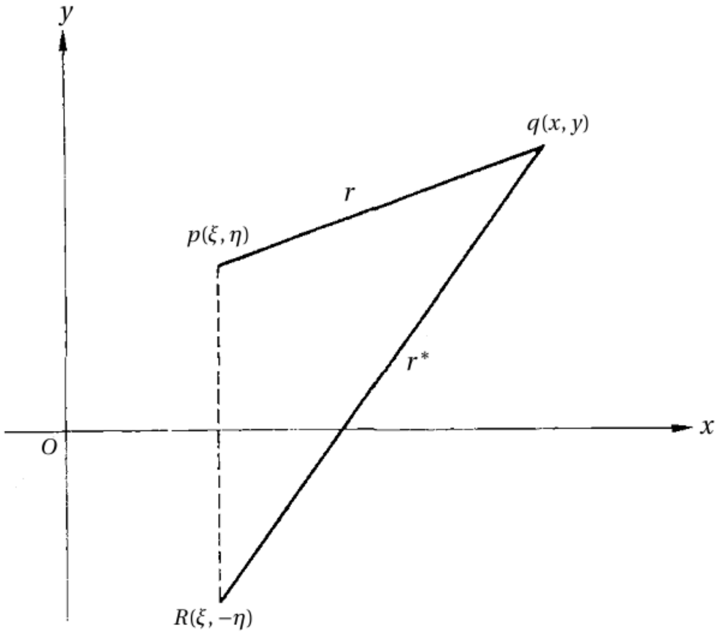


Figure 1.4: Green's functions for the half-plan ($y > 0$).

Consider the reflected image $R : (\xi, -\eta)$ of the point P with respect to the x-axis. It is clear that when Q is located on the boundary $y = 0$ of the half-plane $y \geq 0$, Q is equidistant from P and R. Thus, we take

$$g(x, y; \xi, \eta) = \frac{-1}{2\pi} \ln r^* \tag{10.12}$$

Where

$$r^* = \sqrt{(x - \xi^2) + (y - \eta^2)} \tag{10.13}$$

Clearly, this function is harmonic for all (x,y) in the half-plane $y \geq 0$ and reduce to $-(1/2\pi) \ln r$ on y=0 .Therefor, the Green's function for the half-plane $y \geq 0$ is given by

$$\begin{aligned} G\big(x, y\,;\,\xi, \eta\big) =& \frac{1}{2\pi} \ln(r) - \frac{1}{2\pi} \ln(r^*) = \frac{1}{2\pi} \ln\left(\frac{r}{r^*}\right) \\ =& \frac{1}{2\pi} \ln \frac{\sqrt{\big(x - \xi\big)^2 + \big(y - \eta\big)^2}}{\sqrt{\big(x - \xi\big)^2 + \big(y + \eta\big)^2}} \end{aligned}$$

for the half-plan ($y > 0$).

1.9 Exercises

***Exercise* 1.1.** *Show that the function u defined by* $u(x,y)=\dfrac{1-x^2-y^2}{1-2x+x^2+y^2}$ *is a harmonic function on the disk* $D=\big\{(x,y)\in\mathbb{R}^2,\ x^2+y^2\leq 1\big\}$.

***Exercise* 1.2.** *Determine the solution of the following Laplace equation:*

$$u_{xx}(x,y)+u_{yy}(x,y)=0,\quad 0<x<\pi,\quad 0<y<\pi \tag{1.57}$$

with the following boundary conditions:

$$\begin{cases} u(0,y)=0, & u(\pi,y)=0, \qquad 0\leq y\leq\pi \\ u(x,0)=0, & u(x,\pi)=\sin^3 x, \quad 0\leq x\leq\pi. \end{cases} \tag{1.58}$$

***Exercise* 1.3.** *Using the method of separation of variables, determine the solution of the following Laplace equation:*

$$\Delta u(x,y)=u_{xx}(x,y)+u_{yy}(x,y)=0,\quad 0<x<\pi,\quad 0<y<\pi \tag{1.59}$$

with the following non-homogeneous Neumann boundary conditions:

$$\begin{cases} u_x(0,y)=0, & u_x(\pi,y)=0, \qquad 0\leq y\leq\pi \\ u_y(x,0)=x-\frac{\pi}{2}, & u_y(x,\pi)=0, \quad 0\leq x\leq\pi. \end{cases} \tag{1.60}$$

***Exercise* 1.4.** *Let* $D=\big\{(x,y)\in\mathbb{R}^2,\ x^2+y^2\leq a^2\big\}$ *be a disk of radius a centered at the origin. Let* $C: x^2+y^2=a^2$ *be the circle of center* $(0,0)$ *and radius a.*

Using polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ *with* $0\leq\theta\leq 2\pi$ *and* $0<r\leq a$, *show that the Laplace equation on the disk D, that is,* $\begin{cases}\Delta u(x,y)=u_{xx}(x,y)+u_{yy}(x,y)=0, & \text{in } D,\\ u(x,y)=f(x,y), & \text{on } C=\partial D\end{cases}$ *is equivalent to the following equation:*

$$\Delta w(r,\theta)=w_{rr}(r,\theta)+\frac{1}{r}w_r(r,\theta)+\frac{1}{r^2}w_{\theta\theta}(r,\theta)=0,$$

where $w(r,\theta)=u\big(x(r,\theta),\,y(r,\theta)\big)$ *with the boundary condition* $w(a,\theta)=h(\theta)=f\big(x(r,\theta),\,y(r,\theta)\big)$.

***Exercise* 1.5.** *Using the method of separation of variables, determine u the solution of the fol-*

lowing Laplace equation:

$$\Delta u(x,y) = u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0, \quad \textit{in} \quad D \tag{1.61}$$

where $D = \{0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$ *with the following boundary conditions:*

$$\begin{cases} u(0,y,z) = 0, & u(x,0,z) = 0, \quad u(x,\pi,z) = 0 \\ u(x,y,0) = 0, & u(x,y,\pi) = 0, \quad u(\pi,y,z) = g(y,z). \end{cases} \tag{1.62}$$

where g *is a given function.*

***Exercise* 1.6.** *Solve the following problem:*

$$w_{rr}(r,\theta) + \frac{1}{r} w_r(r,\theta) + \frac{1}{r^2} w_{\theta\theta}(r,\theta) = 0, \quad \textit{for} \quad 0 < r < 1$$

with

$$w(1,\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi. \end{cases}$$

Using the Poisson integral formula, determine $w(0,\theta)$*.*

Chapter 2

Hyperbolic equation

This chapter is devoted to the study the partial differential equation of hyperbolic type. More precisely, we will study the wave equation

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - c^2 \triangle u(\mathbf{x}, t) = F(\mathbf{x}, t),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ are two independent variables with $n \geq 1$, and $c > 0$ is a real number. This PDE naturally appears in many physical problems such as: vibrating strings or membranes, acoustic waves, electromagnetic waves, seismic waves, etc.

2.1 The wave equation in one dimension

We are interested in the second-order partial differential equation in time t and space x of the form:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = F(x, t)$$

where $c > 0$ is a constant. This is the equation of motion of a vibrating string, mathematically referred to as the one-dimensional d'Alembert equation. The constant c will be identified as the wave propagation speed.

Remark 5. *The homogeneous wave equation in spatial dimension n is written as:*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - c^2 \Delta u(\mathbf{x}, t) = 0, & \forall \mathbf{x} \in \mathbb{R}^n,\ t \in \mathbb{R}^+, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n, \\ \frac{\partial u}{\partial t}(\mathbf{x}, 0) = v_0(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^n, \end{cases}$$

where c is the wave speed, and u_0 and v_0 are two given functions in $C^2(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$, respectively.

Definition 2. *The operator*

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

is called the ***wave operator****, or* ***the d'Alembertian*** *in n dimension.*

2.2 d'Alembert's formula for the wave equation

2.2.1 Homogeneous wave equation

We are interested in solving the one-dimensional wave equation defined on the entire real line $\mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}^+, \tag{2.1}$$

with the following initial conditions:

$$u(x,0) = f(x), \quad \text{and} \quad \partial_t u(x,0) = g(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Theorem 4. *The function u defined on $\mathbb{R} \times \mathbb{R}$ by:*

$$u(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \tag{2.3}$$

is a solution to the problem (2.1)–(2.2).

Remark 6. *The formula (2.3) is called d'Alembert's formula for the homogeneous one-dimensional wave equation.*

Proof. We introduce the following change of variables:

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \tag{2.4}$$

Therefore:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}, \\ \dfrac{\partial^2 u}{\partial t^2} = c^2\left[\dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}\right]. \end{cases} \tag{2.5}$$

Hence, using (2.5), the equation (2.1) is equivalent to:

$$-4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

and since $c \neq 0$, we have

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

The general solution of equation (2.6) is of the form

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

therefore:

$$u(x,t) = F(x+ct) + G(x-ct) \tag{2.6}$$

where F and G are arbitrary twice-differentiable functions of a single variable.

Now, we determine F and G such that the solution (2.6) satisfies the initial conditions (2.2). Then, for $t = 0$ in (2.6), we obtain:

$$\begin{cases} F(x) + G(x) = f(x), \\ cF'(x) - cG'(x) = g(x), \end{cases} \qquad \forall x \in \mathbb{R}$$

which implies

$$(S) \begin{cases} F'(x) + G'(x) = f'(x), \\ cF'(x) - cG'(x) = g(x), \end{cases} \qquad \forall x \in \mathbb{R}.$$

Solving the linear system (S) for $F'(x)$ and $G'(x)$ yields:

$$\begin{cases} F'(x) = \frac{1}{2}\left[f'(x) + \frac{1}{c}g(x)\right], \\ G'(x) = \frac{1}{2}\left[f'(x) - \frac{1}{c}g(x)\right]. \end{cases} \tag{2.7}$$

Integrating both equations of (2.7) from 0 to x, we obtain:

$$\begin{cases} F(x) = \dfrac{1}{2}f(x) + \dfrac{1}{2c}\displaystyle\int_0^x g(s)\,ds + c^*, \\ G(x) = \dfrac{1}{2}f(x) - \dfrac{1}{2c}\displaystyle\int_0^x g(s)\,ds - c^*, \end{cases}$$

where c^* is an integration constant.

Now replacing x in the expressions for F and G by $x+ct$ and $x-ct$ respectively, and using (2.6), we obtain:

$$\begin{aligned}
u(x,t) =& F(x+ct)+G(x-ct)\\
=& \frac{1}{2}\Big[f(x-ct)+f(x+ct)\Big]+\frac{1}{2c}\left[\int_{0}^{x+ct} g(s)\,ds-\int_{0}^{x-ct} g(s)\,ds\right]\\
=& \frac{1}{2}\Big[f(x-ct)+f(x+ct)\Big]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.
\end{aligned}$$

This completes the proof of Theorem 4. ☐

Example 4. *Using d'Alembert's formula, determine the solution to equation (2.1) with the following initial conditions:*

$$u(x,0)=\sin x \quad \textit{and} \quad u_t(x,0)=0.$$

Solution: According to formula (2.3), we have

$$u(x,t)=\frac{\sin(x-ct)+\sin(x+ct)}{2}=\sin x\cos(ct),$$

which is a solution of equation (2.1).

Example 5. *Using d'Alembert's formula, solve equation (2.1) with the following initial conditions:*

$$u(x,0)=0 \quad \textit{and} \quad u_t(x,0)=\sin(2x).$$

Solution: According to formula (2.3), we have

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}\sin(2s)\,ds=\frac{\sin(2x)\sin(2ct)}{2c},$$

which is a solution of equation (2.1).

2.2.2 Non homogeneous wave equation

We consider the non-homogeneous wave equation in one dimension defined over the entire real line:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - c^2\frac{\partial^2 u}{\partial x^2}(x,t) = F(x,t), \quad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R}^+, \tag{2.8}$$

with the following initial conditions:

$$u(x,0) = \psi(x), \quad \text{and} \quad \partial_t u(x,0) = \phi(x), \quad -\infty < x < +\infty, \tag{2.9}$$

where F is a source term.

Theorem 5. *Let $F \in C^0(\mathbb{R}\times\mathbb{R}_+)$. Consider the non-homogeneous Cauchy problem (2.8)–(2.9) with $\psi \in C^2(\mathbb{R})$ and $\phi \in C^1(\mathbb{R})$. Then, the problem (2.8)–(2.9) admits a unique solution $u \in C^2(\mathbb{R}\times\mathbb{R}_+)$ given by:*

$$u(x,t) = \frac{1}{2}\left[\psi(x-ct)+\psi(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct}\phi(s)\,ds + \frac{1}{2c}\int\limits_{0}^{t}\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau. \tag{2.10}$$

Remark 7. *The formula (2.10) is called d'Alembert's formula for the non-homogeneous wave equation in one dimension.*

Proof. We introduce the following change of variables:

$$\begin{cases} \xi = x+ct, \\ \eta = x-ct, \end{cases}$$

Then, the PDE (2.8) can be rewritten in the form:

$$\frac{\partial^2 u}{\partial\xi\partial\eta}\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right) = -\frac{1}{4c^2}F\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right). \tag{2.11}$$

Integrating with respect to ξ, we get

$$\begin{aligned}\frac{\partial u}{\partial\eta}\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right) =& \frac{\partial u}{\partial\eta}\left(\frac{\bar\xi+\eta}{2},\frac{\bar\xi-\eta}{2c}\right)\Big|_{\bar\xi=\eta} + \int\limits_{\eta}^{\xi}\frac{\partial^2 u}{\partial\xi\partial\eta}\left(\frac{\bar\xi+\eta}{2},\frac{\bar\xi-\eta}{2c}\right)d\bar\xi \\ =& \frac{1}{2}\frac{\partial u}{\partial x}(\eta,0) - \frac{1}{2c}\frac{\partial u}{\partial t}(\eta,0) - \frac{1}{4c^2}\int\limits_{\eta}^{\xi}F\left(\frac{\bar\xi+\eta}{2},\frac{\bar\xi-\eta}{2c}\right)d\bar\xi.\end{aligned}$$

Integrating this equation over the interval $(\eta,\, \xi)$ gives:

$$\begin{aligned} u(\xi,0) - u\left(\frac{\xi+\eta}{2},\, \frac{\xi-\eta}{2c}\right) =& \underbrace{\frac{1}{2}\int_{\eta}^{\xi} \frac{\partial u}{\partial x}(\bar\eta,0)d\bar\eta}_{I_1} - \frac{1}{2c}\int_{\eta}^{\xi} \frac{\partial u}{\partial t}(\bar\eta,0)d\bar\eta \\ & \underbrace{-\frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\bar\eta}^{\xi} F\left(\frac{\bar\xi+\bar\eta}{2},\, \frac{\bar\xi-\bar\eta}{2c}\right) d\bar\xi d\bar\eta}_{I_2}. \end{aligned} \tag{2.12}$$

Now, in I_1, we have

$$\frac{1}{2}\int_{\eta}^{\xi} \frac{\partial u}{\partial x}(\bar\eta,0)d\bar\eta = \frac{1}{2}\left(u(\xi,0) - u(\eta,0)\right).$$

For I_2, we set:

$$\begin{cases} \bar\xi = \bar x + c\bar t, \\ \bar\eta = \bar x - c\bar t. \end{cases}$$

The integration domain $\eta \le \bar\eta \le \bar\xi \le \xi$ becomes: $\eta \le \bar x - c\bar t \le \bar x + c\bar t \le \xi$, or equivalently:

$$\eta + c\bar t \le \bar x \le \xi - c\bar t, \quad \text{and} \quad 0 \le \bar t \le \frac{1}{2c}(\xi-\eta).$$

The Jacobian determinant of the transformation is:

$$\begin{vmatrix} \frac{\partial\bar\xi}{\partial\bar x} & \frac{\partial\bar\xi}{\partial\bar t} \\ \frac{\partial\bar\eta}{\partial\bar x} & \frac{\partial\bar\eta}{\partial\bar t} \end{vmatrix} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c.$$

Hence,

$$I_2 = \frac{1}{2c}\int_{0}^{(\xi-\eta)/2c}\int_{\eta+c\bar t}^{\xi-c\bar t} F(\bar x,\bar t)\, d\bar x\, d\bar t.$$

Substituting I_1 and I_2 into equation (2.12), we obtain:

$$\begin{aligned} u\left(\frac{\xi+\eta}{2},\, \frac{\xi-\eta}{2c}\right) =& \frac{1}{2}\left(u(\xi,0)+u(\eta,0)\right) + \frac{1}{2c}\int_{\eta}^{\xi} \frac{\partial u}{\partial t}(\bar x,0)\, d\bar x \\ & + \frac{1}{2c}\int_{0}^{(\xi-\eta)/2c}\int_{\eta+c\bar t}^{\xi-c\bar t} F(\bar x,\bar t)\, d\bar x\, d\bar t. \end{aligned}$$

Now, noting that $\xi = x + ct$ and $\eta = x - ct$, and using the initial conditions (2.9), we recover the formula (2.10). □

Example 6. *Using d'Alembert's formula, determine the solution of the equation* (2.8) *with* $F(x,t) = x$, $c = 1$, *and the following initial conditions:*

$$u(x,0) = u_t(x,0) = 0.$$

Solution: According to the formula (2.10), we have:

$$u(x,t) = \frac{1}{2}\int\limits_0^t \int\limits_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi \, d\tau.$$

We calculate the first integral:

$$\int\limits_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2}\left[\xi^2\right]_{x-(t-\tau)}^{x+(t-\tau)} = \frac{1}{2}\left[(x+(t-\tau))^2 - (x-(t-\tau))^2\right].$$

Then

$$\int\limits_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2}\left[\left(x^2 + 2x(t-\tau) + (t-\tau)^2\right) - \left(x^2 - 2x(t-\tau) + (t-\tau)^2\right)\right].$$

The terms in x^2 and $(t-\tau)^2$ cancel out, and we obtain

$$\int\limits_{x-(t-\tau)}^{x+(t-\tau)} \xi \, d\xi = \frac{1}{2}\left[4x(t-\tau)\right].$$

Thus, we have:

$$u(x,t) = \frac{1}{2}\int\limits_0^t 4x(t-\tau)\, d\tau = 2x\int\limits_0^t (t-\tau)\, d\tau.$$

The integral is straightforward to calculate:

$$\int\limits_0^t (t-\tau)\, d\tau = \frac{t^2}{2}.$$

Therefore, the solution is:

$$u(x,t) = 2x \times \frac{t^2}{2} = xt^2.$$

Thus, the solution to the equation (2.8) is:

$$u(x,t) = \frac{xt^2}{2}.$$

2.3 One dimensional damped wave equation in a bounded domain

In this subsection, we will use the separation of variables method to determine the solution of the damped one-dimensional wave equation in a bounded domain. The mathematical model is written as follows:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + hu(x,t) = 0, \qquad 0 < x < L, \ \ t > 0 \tag{2.13}$$

where h is a positive constant, with the following boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \tag{2.14}$$

and the initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad 0 \leq x \leq L. \tag{2.15}$$

Using the method of separation of variables, we assume that any non-trivial solution is of the form:

$$u(x,t) = X(x)T(t) \tag{2.16}$$

which satisfies all the homogeneous boundary conditions in (2.14) and the homogeneous initial condition in (2.15). We have disregarded the non-homogeneous condition $u(x,0) = f(x)$.

Substituting this solution into the PDE (2.13), we obtain:

$$X(x)T''(t) - c^2 X''(x)T(t) + hX(x)T(t) = 0 \tag{2.17}$$

where X'' is the second derivative of X with respect to x, and T'' is the second derivative of T

with respect to t. Dividing this equation by $c^2X(x)T(t)$, we get:

$$\frac{T''(t)}{c^2T(t)} - \frac{X''(x)}{X(x)} + \frac{h}{c^2} = 0 \Longrightarrow \frac{T''(t)}{c^2T(t)} + \frac{h}{c^2} = \frac{X''(x)}{X(x)}. \tag{2.18}$$

Now in (2.18), the left-hand side is a function of t only, while the right-hand side is a function of x only. Therefore, both expressions must be constant and we can write:

$$\frac{T''(t)}{c^2T(t)} + \frac{h}{c^2} = \frac{X''(x)}{X(x)} = -\lambda, \quad \lambda \in \mathbb{R}. \tag{2.19}$$

Also, from the boundary conditions (2.14) and the homogeneous initial condition in (2.15), we obtain:

$$\begin{cases} u(0,t) = 0 \Longrightarrow X(0)T(t) = 0, \quad x \in [0,L] \Longrightarrow X(0) = 0, \\ u(L,t) = 0 \Longrightarrow X(L)T(t) = 0, \quad x \in [0,L] \Longrightarrow X(L) = 0, \\ u_t(x,0) = 0 \Longrightarrow X(x)T'(0) = 0, \quad t > 0 \Longrightarrow T'(t) = 0. \end{cases} \tag{2.20}$$

In summary, we are led to study the following problems:

$$(\mathcal{P}_1) \begin{cases} X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \\ X(0) = 0, \ X(L) = 0 \end{cases}$$

and

$$(\mathcal{P}_2) \begin{cases} T''(t) + \big(\lambda c^2 + h\big)T(t) = 0, \quad t > 0, \\ T'(0) = 0. \end{cases}$$

Solution of problem $(\mathcal{P}_1)$

We distinguish three possible cases for λ.

<u>Case 1</u>: If $\lambda = -k^2 < 0$, where k is a non-zero real number. Then, the problem $(\mathcal{P}_1)$ admits a general solution of the form:

$$X(x) = c_1e^{-kx} + c_2e^{kx}$$

where c_1 and c_2 are constants. From the boundary conditions $X(0) = 0, \ X(L) = 0$, we obtain the following system:

$$(S) \begin{cases} c_1 + c_2 = 0, \\ c_1e^{-kL} + c_2e^{kL} = 0. \end{cases}$$

The system (S) has only one solution, which is the trivial solution $(c_1, c_2) = (0, 0)$. Indeed, the determinant of the system (S) is:

$$\begin{vmatrix} 1 & 1 \\ e^{-kL} & e^{kL} \end{vmatrix} = e^{kL} - e^{-kL} = \frac{e^{2kL}-1}{e^{kL}} \neq 0.$$

Otherwise,

$$e^{2kL} - 1 = 0 \Longrightarrow e^{2kL} = 1 \Longrightarrow 2kL = 0.$$

But this is absurd since k and L are non-zero. Therefore, we can conclude that:

$$\begin{pmatrix} 1 & 1 \\ e^{-kL} & e^{kL} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

since the matrix $\begin{pmatrix} 1 & 1 \\ e^{-kL} & e^{kL} \end{pmatrix}$ is invertible. Therefore, in the case where $\lambda = -k^2 < 0$, we find that $X(x) \equiv 0$ and consequently $u(x,t) \equiv 0$ for all $0 < x < L$. Thus, we must exclude the case $\lambda < 0$.

<u>Case 2</u>: If $\lambda = 0$, then $X(x) = c_1 x + c_2$, where c_1 and c_2 are arbitrary real numbers. However, from the conditions $X(L) = 0$ and $X(0) = 0$, we obtain:

$$\begin{cases} c_2 = 0, \\ c_1 L + c_2 = 0 \end{cases} \Longrightarrow \begin{cases} c_2 = 0, \\ c_1 = 0 \end{cases} \quad \text{since} \quad L \neq 0.$$

Therefore, in the case where $\lambda = 0$, we find that $X(x) \equiv 0$ and consequently $u(x,t) \equiv 0$ for all $0 < x < L$. Thus, we must also exclude the case $\lambda = 0$.

<u>Case 3</u>: If $\lambda > 0$, the general solution to problem $(\mathcal{P}_1)$ is of the form:

$$X(x) = c_1 \sin\big(\sqrt{\lambda} x\big) + c_2 \cos\big(\sqrt{\lambda} x\big)$$

where c_1 and c_2 are arbitrary real numbers. From the condition $X(0) = 0$, we obtain $c_2 = 0$. Therefore, a non-trivial solution takes the form:

$$X(x) = c_1 \sin\big(\sqrt{\lambda} x\big) \quad \text{with } c_1 \neq 0.$$

Also, from the condition $X(L) = 0$, we get $\sin\big(\sqrt{\lambda}L\big) = 0$, hence $\sqrt{\lambda}L = n\pi$, with $n \in \mathbb{Z}$. Since $\lambda > 0$, we finally obtain the eigenvalues of problem $(\mathcal{P}_1)$:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \text{with } n \in \mathbb{Z}$$

and the eigenfunctions are of the form:

$$X_n(x) = c_1^n \sin\Big(\frac{n\pi x}{L}\Big), \quad \text{with } n \in \mathbb{Z},$$

where c_1^n are constants. Since $\lambda > 0$, we restrict to the case $n \in \mathbb{N}^*$ (positive integers), so:

$$X_n(x) = c_1^n \sin\Big(\frac{n\pi x}{L}\Big), \quad \text{with } n \in \mathbb{N}^*. \tag{2.21}$$

We now turn to problem $(\mathcal{P}_2)$.

For $n \geq 1$, $n \in \mathbb{N}$, and with $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$, the general solution is of the form:

$$T_n(t) = c_3 \sin\big(t\sqrt{\lambda_n c^2 + h}\big) + c_4 \cos\big(t\sqrt{\lambda_n c^2 + h}\big)$$

where c_3 and c_4 are arbitrary constants. From the condition $T_n'(0) = 0$, we get $c_3 = 0$, so:

$$T_n(t) = c_4^n \cos\left(t\sqrt{\left(\frac{n\pi c}{L}\right)^2 + h}\right), \quad \text{with } n \in \mathbb{N}^*. \tag{2.22}$$

Finally, from (2.21) and (2.22), the general solution of the PDE (2.13) with the boundary conditions (2.14) and initial conditions (2.15) is given by:

$$\begin{aligned} u_n(x,t) = X_n(x)T_n(t) &= c_1^n \sin\left(\frac{n\pi x}{L}\right) \cdot c_4^n \cos\left(t\sqrt{\left(\frac{n\pi c}{L}\right)^2 + h}\right), \quad n \in \mathbb{N}^* \\ &= C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(t\sqrt{\left(\frac{n\pi c}{L}\right)^2 + h}\right), \quad n \in \mathbb{N}^* \end{aligned} \tag{2.23}$$

with $C_n = c_1^n c_4^n$. Using the principle of superposition, we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(t\sqrt{\left(\frac{n\pi c}{L}\right)^2 + h}\right) \tag{2.24}$$

which is also a solution of the problem (2.13)–(2.15), where C_n are arbitrary constants.

Now, we determine the constants C_n such that the solution (2.24) satisfies the non-homogeneous initial condition $u(x, 0) = f(x)$ for all $0 \leq x \leq L$. Setting $t = 0$ in (2.24), we obtain:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right), \quad x \in [0, L]. \tag{2.25}$$

This expansion is called the generalized Fourier series of the function f with respect to the eigenfunctions (2.21), and the C_n are the Fourier coefficients of this series. To determine the C_n, for a fixed $m \in \mathbb{N}^*$, we multiply equation (2.25) by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate term by term over $[0, L]$, giving:

$$\int\limits_0^L \sin\left(\frac{m\pi x}{L}\right) f(x)dx = \sum_{n=1}^{\infty} C_n \int\limits_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Using the fundamental orthogonality property of the eigenfunctions: if n and m are non-zero integers, then

$$\int\limits_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } n = m. \end{cases}$$

In the first case ($n \neq m$), we say that the functions $X_m(x) = \sin\left(\frac{m\pi x}{L}\right)$ and $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ are orthogonal over the interval $[0, L]$.

Finally, using this property, we obtain:

$$C_n = \frac{\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx} = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx,$$

for all $n = 1, 2, \ldots$

Example 7. *Solve the problem* (2.13) *with* $L = \pi$, $h = 1$, $c = 1$, *and:*

$$f(x) = \begin{cases} x & 0 \leq x \leq \pi/2, \\ \pi - x & \pi/2 \leq x \leq \pi. \end{cases}$$

Solution: From above, we have:

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin(nx) \cos\left(t\sqrt{n^2 + 1}\right),$$

with

$$\begin{aligned}C_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)f(x)dx\\ &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin(nx)dx\\ &= \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[-(\pi - x)\frac{\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi}\\ &= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right).\end{aligned}$$

Now, observe that $\sin\left(\frac{n\pi}{2}\right) = 0$ when $n = 2k$, and $\sin\left(\frac{n\pi}{2}\right) = (-1)^{k+1}$ when $n = 2k-1$ for $k = 1, 2, \ldots$

Therefore, the solution of problem (2.13) is:

$$u(x,t) = \frac{4}{\pi}\sum_{k=1}^{\infty} \sin\big((2k-1)x\big)\cos\left(\sqrt{(2k-1)^2+1}\ t\right).$$

2.4 Representation of the solution

In this section, we provide an explicit representation of the solution to the wave equation in dimensions $n = 2$ and $n = 3$.

2.4.1 Wave equation in three dimension

Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and consider the wave equation in $\mathbb{R}^3$ with initial conditions. The mathematical problem is written as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\mathbf{x},t) - c^2\Delta u(\mathbf{x},t) = 0, & \forall \mathbf{x}\in\mathbb{R}^3,\ \ t\in\mathbb{R}^+,\\ u(\mathbf{x},0) = g(\mathbf{x}), & \forall \mathbf{x}\in\mathbb{R}^3,\\ \frac{\partial u}{\partial t}(\mathbf{x},0) = h(\mathbf{x}), & \forall \mathbf{x}\in\mathbb{R}^3, \end{cases} \tag{2.26}$$

where c is the wave speed, and g and h are given functions.

First we need a lemma that reduces the problem to the case $g = 0$ (and which actually holds in any dimension). Let us denote by w_h the solution of the following problem:

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(\mathbf{x},t) - c^2\Delta w(\mathbf{x},t) = 0, & \forall \mathbf{x}\in\mathbb{R}^3,\ \ t\in\mathbb{R}^+,\\ w(\mathbf{x},0) = 0,\quad \frac{\partial w}{\partial t}(\mathbf{x},0) = h(\mathbf{x}), & \forall \mathbf{x}\in\mathbb{R}^3. \end{cases} \tag{2.27}$$

Lemma 2. *If $w_g \in C^3\big(\mathbb{R}^3 \times [0,+\infty)\big)$, then $v = \partial_t w_g$ solves the problem*

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}(\mathbf{x},t) - c^2 \Delta v(\mathbf{x},t) = 0, & \forall \mathbf{x} \in \mathbb{R}^3, \quad t \in \mathbb{R}^+, \\ v(\mathbf{x},0) = g(\mathbf{x}), \quad \frac{\partial v}{\partial t}(\mathbf{x},0) = 0, & \forall \mathbf{x} \in \mathbb{R}^3. \end{cases} \tag{2.28}$$

Therefore the solution of (2.26) is given by:

$$u(\mathbf{x},t) = \partial_t w_g(\mathbf{x},t) + w_h(\mathbf{x},t). \tag{2.29}$$

Proof. Let $v = \partial_t w_g$. Differentiating the wave equation with respect to t, we obtain:

$$0 = \partial_t \left(w_{tt}(\mathbf{x},t) - c^2 \Delta w(\mathbf{x},t) \right) = \left(\partial_{tt} - c^2 \Delta \right) \partial_t w_g(\mathbf{x},t) = v_{tt}(\mathbf{x},t) - c^2 \Delta v(\mathbf{x},t).$$

Moreover,

$$v(\mathbf{x},0) = \partial_t w_g(\mathbf{x},0) = g(\mathbf{x}) \quad \text{and} \quad v_t(\mathbf{x},0) = \partial_{tt} w_g(\mathbf{x},0) = c^2 \Delta w_g(\mathbf{x},0) = 0.$$

Therefore, v is a solution of problem (2.28), and $u = v + w_h$ is a solution of (2.26). □

Theorem 6. (***Kirchhoff's formula***)**.** *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then the unique solution $u \in C^2(\mathbb{R}^3 \times \mathbb{R}^+)$ of problem (2.26) is given by:*

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \int_{\partial B(\mathbf{x},ct)} g(\sigma)\, d\sigma \right] + \frac{1}{4\pi c^2 t} \int_{\partial B(\mathbf{x},ct)} h(\sigma)\, d\sigma, \tag{2.30}$$

where $B(\mathbf{x},ct)$ is the ball of center $\mathbf{x}$ and radius ct.

2.4.2 Wave equation in two dimension

Using Kirchhoffs formula, we can obtain the solution to the Cauchy problem for the wave equation in two dimensions. More precisely, we have the following result:

Theorem 7. (***Poisson's Formula***)**.** *Let $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then*

$$u(\mathbf{x},t) = \frac{1}{2\pi c}\left\{ \frac{\partial}{\partial t} \int_{B(\mathbf{x},ct)} \frac{g(\mathbf{y})\, d\mathbf{y}}{\sqrt{c^2t^2 - |\mathbf{x}-\mathbf{y}|^2}} + \int_{B(\mathbf{x},ct)} \frac{h(\mathbf{y})\, d\mathbf{y}}{\sqrt{c^2t^2 - |\mathbf{x}-\mathbf{y}|^2}} \right\} \tag{2.31}$$

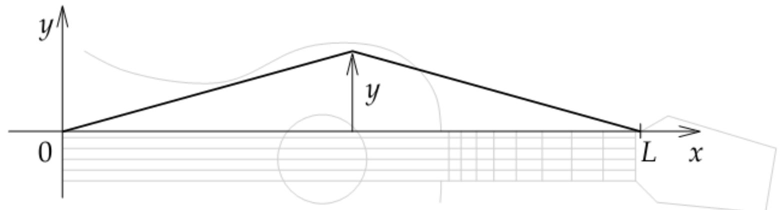
is the unique solution $u \in C^2(\mathbb{R}^2 \times \mathbb{R}^+)$ of the following problem:

$$\begin{cases} u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = 0, & \forall \mathbf{x} \in \mathbb{R}^2,\ t \in \mathbb{R}^+, \\ u(\mathbf{x},0) = g(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^2, \\ u_t(\mathbf{x},0) = h(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^2, \end{cases}$$

where c is the wave speed, and B denotes the disk centered at $\mathbf{x}$ with radius ct.

2.5 Vibrating strings

Imagine a tensioned guitar string of length L that can vibrate. We will only consider vibrations in one direction. Let x denote the position along the string, let t denote time, and let $y(x,t)$ denote the displacement of the string from the rest position. See



Vibrating string of length L, x is position, y is displacement.

This string is subjected to an external force with linear density $f(x,t)$. We assume that the displacements are sufficiently small so that the tension remains constant over time, which allows us to linearize the fundamental equations of dynamics applied to a small portion of the string. The equation that governs this setup is the so-called *one-dimensional wave equation*:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) + f(x,t),$$

for some constant $a > 0$.

Boundary conditions: since the two ends of the string are fixed at $x = 0$ and $x = L$, the displacement function at these points must be zero, i.e.:

$$u(0,t) = 0, \quad \text{and} \quad u(L,t) = 0, \quad t \in \mathbb{R}^+.$$

Initial conditions: since the PDE is second order in time, we must impose two initial conditions on $u(x,t)$ and $\partial_t u(x,t)$ at time $t = 0$, namely:

$$u(x,0) = h(x), \quad \text{and} \quad \partial_t u(x,0) = g(x), \quad x \in [0,L],$$

where h and g are given functions.

2.5.1 Solving the vibrating string equation by the Fourier method

Suppose an elastic string with both ends fixed at points 0 and $L = 1$ m, subjected to an external linear force of the form

$$F(x,t) = \sin(m\pi x)\sin(\omega t)$$

for all $m \in \mathbb{N}$ and $\omega^2 \neq m^2\pi^2$. Let $u(x,t)$ denote the displacement from equilibrium at position x and time t. The evolution equation for u, with boundary and initial conditions, reads:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = \sin(m\pi x)\sin(\omega t), & 0 < x < 1, \quad t \geq 0, \\ u(0,t) = u(1,t) = 0, & t \geq 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \\ u_t(x,0) = 0, & 0 \leq x \leq 1, \end{cases} \tag{2.32}$$

with wave speed $c = 1$ m/s.

We seek a non-trivial solution of the corresponding homogeneous problem in the separated variables form

$$y(x,t) = X(x)T(t).$$

Substituting this form into equation (2.32) leads to the Sturm-Liouville problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X(0) = X(1) = 0, \end{cases}$$

where $\lambda \in \mathbb{R}$. The eigenvalues of this problem are $\lambda_n = n^2\pi^2$, with $n \in \mathbb{N}^*$, and the corresponding eigenfunctions are

$$X_n(x) = \sin(n\pi x),$$

(see the solution of problem $(\mathcal{P}_1)$ on page 15 for more details). Thus, the general solution of problem (2.32) can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x), \tag{2.33}$$

where the coefficients $T_n(t)$ are the Fourier coefficients of the function $u(\cdot,t)$. Our goal is to determine these coefficients.

To find $T_n(t)$, we formally substitute the series (2.33) into equation (2.32) and differentiate term-by-term, yielding

$$\sum_{n=1}^{\infty} \big(T_n''(t) + n^2\pi^2 T_n(t)\big)\sin(n\pi x) = \sin(m\pi x)\sin(\omega t).$$

By orthogonality of the sine functions, for $n = m$, we obtain the ordinary differential equation

$$T_m''(t) + m^2\pi^2 T_m(t) = \sin(\omega t). \tag{2.34}$$

Given the zero initial conditions, the solution to problem (2.34) is

$$T_m(t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right).$$

For $n \neq m$, the corresponding ODE is homogeneous:

$$T_n''(t) + n^2\pi^2 T_n(t) = 0,$$

with zero initial conditions, which implies

$$T_n(t) = 0, \quad \text{for } n \neq m.$$

Finally, the solution to problem (2.32) is

$$u(x,t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right)\sin(m\pi x).$$

2.6 Vibrating plate

The transverse vibrations of a plate (or a beam) are described by the following partial differential equation:

$$\rho\, u_{tt}(x,t) + EI\, u_{xxxx}(x,t) = 0, \quad 0 < x < l, \quad t \in \mathbb{R}^+, \tag{2.35}$$

where l is the length of the beam and $u(x,t)$ represents the transverse displacement of the beam at position x and time t. The constants ρ, E, and I denote respectively the mass per unit length, the modulus of elasticity, and the moment of inertia of the beam's cross-section.

The boundary conditions at the ends of the beam generally depend on the type of support and are typically one of the following:

(a) A **fixed end** also known as **built-in** or a clamped end has its displacement and slope equal to zero (see Figure 2.1-[a]): the boundary conditions are then

$$u(l,t) = u_x(l,t) = 0, \quad t \in \mathbb{R}^+. \tag{2.36}$$

(b) A **simply supported end** has displacement and moment equal to zero (see Figure 2.1-[b]): the boundary conditions become

$$u(l,t) = 0, \quad u_{xx}(l,t) = 0, \quad t \in \mathbb{R}^+. \tag{2.37}$$

(c) A **free end** has zero moment and zero shear (see Figure 2.1-[c]): in this case, the conditions are:

$$u(0,t) = 0, \quad u(l,t) = 0, \quad u_{xx}(0,t) = 0, \quad u_{xx}(l,t) = 0, \quad t \in \mathbb{R}^+. \tag{2.38}$$

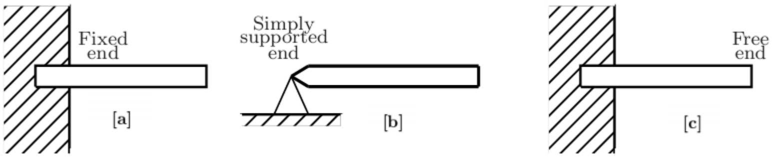


Figure 2.1: Geometry of the beam with different types of supports.

Initial conditions: Suppose that the initial displacement and initial velocity of the beam are known, that is:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{2.39}$$

where f and g are given functions defined for all $x \in]0,l[$.

2.6.1 Separation of variables method for the vibrating plate equation

According to the method of separation of variables, the solution of problem (2.35) with the boundary conditions (2.38) and the initial conditions (2.39) with $g(x) \equiv 0$ is given by:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{c\,n^2\,\pi^2\,t}{l^2}\right) \quad \text{where} \quad \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \tag{2.40}$$

is the odd Fourier series of the function f on the interval $[0,l]$ with $c = \sqrt{\frac{EI}{\rho}}$.

Indeed, consider the following problem:

$$(*) \quad \begin{cases} u_{tt}(x,t) - c^2 u_{xxxx}(x,t) = 0, & 0 < x < l,\ t \in \mathbb{R}^+, \\ u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, & t \geq 0, \\ u(x,0) = f(x), & 0 \leq x \leq l, \\ u_t(x,0) = 0, & 0 \leq x \leq l. \end{cases}$$

We will now determine non-trivial solutions of $(*)$ of the form $u(x,t) = X(x)T(t)$. Substituting this solution into the PDE, we obtain

$$X(x)T^{(2)}(t) + c^2 X^{(4)}(x)T(t) = 0, \tag{2.41}$$

where $X^{(4)}$ is the fourth derivative of X with respect to x and $T^{(2)}$ is the second derivative of T with respect to t.

Dividing both sides of equation (2.41) by $X(x)T(t)$, we get:

$$\frac{X^{(4)}(x)}{X(x)} = -\frac{1}{c^2}\frac{T^{(2)}(t)}{T(t)},$$

for all $(x,t) \in (0,l) \times [0,\infty)$. The right-hand side of this equation is a function of t, and the left-hand side is a function of x. For the equality to hold, each side must be constant. Thus,

$$\frac{X^{(4)}(x)}{X(x)} = -\frac{1}{c^2}\frac{T^{(2)}(t)}{T(t)} = \lambda, \quad \lambda \in \mathbb{R}.$$

We then have a system of two ordinary differential equations:

$$\frac{d^4X(x)}{dx^4} - \lambda X(x) = 0 \quad \text{and} \quad \frac{d^2T(t)}{dt^2} + \lambda c^2 T(t) = 0.$$

We also have the following conditions:

$$\begin{cases} u(0,t)=0 \Longrightarrow X(0)T(t)=0, \quad x\in[0,l] \Longrightarrow X(0)=0, \\ u(l,t)=0 \Longrightarrow X(l)T(t)=0, \quad x\in[0,l] \Longrightarrow X(l)=0, \\ u_{xx}(0,t)=0 \Longrightarrow X''(0)T(t)=0, \quad x\in[0,l] \Longrightarrow X''(0)=0, \\ u_{xx}(l,t)=0 \Longrightarrow X''(l)T(t)=0, \quad x\in[0,l] \Longrightarrow X''(l)=0, \\ u_t(x,0)=0 \Longrightarrow X(x)T'(0)=0, \quad t\in[0,\infty) \Longrightarrow T'(0)=0. \end{cases}$$

We must therefore consider the different possible cases for λ.

Case 1: if $\lambda=-k^4<0$, with $k>0$. Then, the ODE $\frac{d^4X(x)}{dx^4}-\lambda X(x)=0$ admits a general solution of the form:

$$\begin{aligned} X(x) =&A\exp\left(\frac{\sqrt{2}}{2}kx\right)\cos\left(\frac{\sqrt{2}}{2}kx\right)+B\exp\left(\frac{\sqrt{2}}{2}kx\right)\sin\left(\frac{\sqrt{2}}{2}kx\right) \\ &+C\exp\left(-\frac{\sqrt{2}}{2}kx\right)\cos\left(\frac{\sqrt{2}}{2}kx\right)+D\exp\left(-\frac{\sqrt{2}}{2}kx\right)\cos\left(\frac{\sqrt{2}}{2}kx\right) \end{aligned}$$

where A, B, C, and D are arbitrary constants. Considering the conditions $X(0)=0$, $X(l)=0$, $X''(0)=0$, $X''(l)=0$, we get the following system of linear equations:

$$M\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with

$$M=\begin{pmatrix} 1 & 0 & 1 & 0 \\ e^{kl/\sqrt{2}}\cos\left(kl/\sqrt{2}\right) & e^{kl/\sqrt{2}}\sin\left(kl/\sqrt{2}\right) & e^{-kl/\sqrt{2}}\cos\left(kl/\sqrt{2}\right) & e^{-kl/\sqrt{2}}\sin\left(kl/\sqrt{2}\right) \\ 0 & k^2 & 0 & -k^2 \\ -k^2e^{kl/\sqrt{2}}\sin\left(kl/\sqrt{2}\right) & k^2e^{kl/\sqrt{2}}\cos\left(kl/\sqrt{2}\right) & k^2e^{kl/\sqrt{2}}\sin\left(kl/\sqrt{2}\right) & -k^2e^{kl/\sqrt{2}}\sin\left(kl/\sqrt{2}\right) \end{pmatrix}.$$

But, we have: $\det(M)=2k^4\Big(\cosh\big(\sqrt{2}\,kl\big)-\cos\big(\sqrt{2}\,kl\big)\Big)\neq 0$ since: recall that $\cosh(x)\geq 1$ with $\cosh(x)=1 \Longleftrightarrow x=0$ and $\cos(x)\leq 1$ with $\cos(x)=1 \Longleftrightarrow x=2\pi n$ if $n\in\mathbb{Z}$.

Therefore, the above linear system admits only the trivial solution: $(A,B,C,D)=(0,0,0,0)$, so $X(x)\equiv 0$ and $u(x,t)\equiv 0$ for all $x\in[0,l]$. We must exclude the case $\lambda<0$.

Case 2: if $\lambda=0$, the solution of the ODE $\frac{d^4X(x)}{dx^4}=0$ is $X(x)=A+Bx+Cx^2+Dx^3$ where A,

B, C, and D are arbitrary constants. Considering the conditions $X(0) = 0,\ X(l) = 0,\ X''(0) = 0,\ X''(l) = 0$, we get the following system of linear equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6l \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The above linear system has only the trivial solution
$(A, B, C, D) = (0, 0, 0, 0)$, so $X(x) \equiv 0$ and $u(x, t) \equiv 0$ for all $x \in [0, l]$. We must exclude the case $\lambda = 0$.

Case 3: if $\lambda = k^4$ with $k > 0$, the solution of the ODE $\frac{d^4X(x)}{dx^4} - k^4X(x) = 0$ is:

$$X(x) = A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx)$$

where A, B, C, and D are arbitrary constants. According to the conditions $X(0) = 0,\ X(l) = 0,\ X''(0) = 0,\ X''(l) = 0$, we obtain the following system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ \cos(kl) & \sin(kl) & \cosh(kl) & \sinh(kl) \\ -k^2 & 0 & k^2 & 0 \\ -k^2\cos(kl) & -k^2\sin(kl) & k^2\cosh(kl) & k^2\sinh(kl) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We thus obtain $A = C = D = 0$ and $B\sin(kl) = 0$. Since we want non-trivial solutions, we can assume $B \neq 0$ and $\sin(kl) = 0$. Therefore,

$$k = \frac{n\pi}{l} \quad \text{and} \quad \lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^4, \quad n \in \mathbb{N}^*.$$

Thus, for $\lambda = \lambda_n$ the solution of the ODE $\frac{d^4X(x)}{dx^4} + \lambda_nX(x) = 0$ with the conditions $X(0) = 0,\ X(l) = 0,\ X''(0) = 0,\ X''(l) = 0$ is:

$$X_n(x) = B_n\sin\left(\frac{n\pi x}{l}\right), \quad (n = 1, 2, ...). \tag{2.42}$$

where B_n are arbitrary constants.

Now, considering the equation $T_n^{(2)}(t)+\lambda_n T_n(t)=0$, the general solution is

$$T_n(t)=A'\cos\left(\frac{cn^2\pi^2\,t}{l}\right)+B'\sin\left(\frac{cn^2\pi^2\,t}{l}\right),\quad (n=1,2,...)$$

where A' and B' are arbitrary constants. Since $T_n'(0)=0$, it follows that $B'=0$ and the desired solution is:

$$T_n(t)=A'\cos\left(\frac{cn^2\pi^2\,t}{l}\right),\quad (n=1,2,...). \tag{2.43}$$

Hence, according to (2.42) and (2.43), the solution of problem $(*)$ is:

$$u_n(x,t)=X_n(x)T_n(t)=a_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{cn^2\pi^2\,t}{l}\right),\quad (n=1,2,...)$$

where a_n are constants.

Due to the linearity of the PDE, by the superposition principle, we get that:

$$u_n(x,t)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{cn^2\pi^2\,t}{l}\right) \tag{2.44}$$

is a solution of $(*)$. Returning now to the initial problem, we want:

$$\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{l}\right)=f(x).$$

Therefore, the a_n are the coefficients of the odd Fourier series of the function f, i.e.:

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx,\quad \text{for}\quad n\geq 1.$$

2.7 Exercises

***Exercise* 2.1.** *Show that the function u defined by $u(x,t)=t\sin\pi x$ is a solution to the following boundary value problem:*

$$(P)\begin{cases} u_{tt}(x,t)-u_{xx}(x,t)=\pi^2 t\sin\pi x & 0<x<1;\ t>0\\ u(x,0)=0,\quad u_t(x,0)=\sin\pi x & 0\leq x\leq 1\\ u(0,t)=0,\quad u(1,t)=0 & t\geq 0.\end{cases}$$

***Exercise* 2.2.** *Using d'Alembert's formula, determine the solution of the following non-homogeneous problem:*

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = t, & -\infty < x < +\infty;\ t > 0, \\ u(x,0) = x, & -\infty < x < +\infty, \\ u_t(x,0) = 0, & -\infty < x < +\infty. \end{cases}$$

***Exercise* 2.3.** *Solve the following non-homogeneous wave equation using d'Alembert's formula:*

$$\begin{cases} u_{tt}(x,t) - 9u_{xx}(x,t) = e^{x} - e^{-x}, & -\infty < x < +\infty;\ t > 0, \\ u(x,0) = x, & -\infty < x < +\infty, \\ u_t(x,0) = \sin x, & -\infty < x < +\infty. \end{cases}$$

***Exercise* 2.4.** *We consider the following problem:*

$$(\mathcal{H}) \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = \beta \sin\left(\frac{\pi x}{L}\right), & 0 < x < L;\ t > 0, \\ u_t(x,0) = 0; \quad u(x,0) = 0, & 0 \leq x \leq L, \\ u(0,t) = u(L,t) = 0, & t > 0, \end{cases}$$

where β is a constant.

1) Determine a function $u_p(x)$, independent of t, that solves $(\mathcal{H})$.

2) Let $u(x,t) = v(x,t) + u_p(x)$. Show that v satisfies the following problem:

$$(\mathcal{H}_p) \begin{cases} v_{tt}(x,t) = v_{xx}(x,t), & 0 < x < L;\ t > 0, \\ v_t(x,0) = 0, & 0 \leq x \leq L, \\ v(x,0) = -u_p(x), & 0 \leq x \leq L, \\ v(0,t) = v(L,t) = 0, & t > 0. \end{cases}$$

3) Using the method of separation of variables, solve the problem $(\mathcal{H}_p)$, and deduce u, the solution to $(\mathcal{H})$.

Chapter 3

Parabolic equation

3.1 Introduction

Partial differential equations of parabolic type are used to study heat conduction and diffusion processes. The simplest parabolic equation is the heat equation with a source term F:

$$\frac{\partial u}{\partial t}(\mathbf{x},t) - k\Delta_{\mathbf{x}} u(\mathbf{x},t) = F(\mathbf{x},t), \quad \text{where} \quad \Delta_{\mathbf{x}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

where k is a physical constant, $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ are two independent variables and $n \geq 1$. This equation is first-order in time and second-order in space.

The heat equation models evolutionary phenomena: heat diffusion, spread of chemical substances, mixing of species, etc.

3.2 One dimensional heat equation

3.2.1 Case of a finite rod: Physical model

The temperature $u(x,t)$ at point x at time t in a rod of length L, with density ρ, specific heat c, and thermal conductivity μ, evolves over time according to the equation:

$$c\rho\frac{\partial u}{\partial t}(x,t) - \mu\frac{\partial^2 u}{\partial x^2}(x,t) = F(x,t), \quad (x,t) \in (0,L) \times \mathbb{R}^+. \tag{3.1}$$

where F is the heat source in the rod. This problem, first-order in time, is the model of parabolic problems.

To determine the solution, we need to fix an **initial condition** in time: the value of the temperature u at time $t = 0$ is:

$$u(x, 0) = f(x), \quad 0 \leq x \leq L. \tag{3.2}$$

This is called an initial value problem or Cauchy problem.

Moreover, different boundary conditions can be imposed at the ends to fully determine the solution. These boundary conditions are often of the form:

Dirichlet condition

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad t \geq 0. \tag{3.3}$$

Neumann condition

$$-\frac{\partial u}{\partial x}(0, t) = h_1(t), \quad \frac{\partial u}{\partial x}(L, t) = h_2(t), \quad t \geq 0. \tag{3.4}$$

Robin condition

$$\begin{aligned} &-\frac{\partial u}{\partial x}(0, t) + \alpha u(0, t) = h_1(t), \\ &\frac{\partial u}{\partial x}(L, t) + \alpha u(L, t) = h_2(t), \quad t \geq 0 \end{aligned} \tag{3.5}$$

where α is a constant. In the case where $h_1 = h_2 = 0$, the boundary conditions are said to be homogeneous.

3.2.2 Solving the homogeneous problem by the separation of variables method

In this subsection, we focus on solving the one-dimensional heat equation with **homogeneous Dirichlet boundary conditions** using the method of separation of variables. We consider the following initial problem:

$$u_t(x, t) = k u_{xx}(x, t), \quad 0 < x < L, \quad t > 0 \tag{3.6}$$

where $k = \frac{\mu}{c\rho}$, with boundary conditions:

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0 \tag{3.7}$$

and initial condition:

$$u(x,0) = f(x), \quad x \in [0, L]. \tag{3.8}$$

By separation of variables, we seek a solution of (3.6) of the form:

$$u(x,t) = X(x)T(t), \quad \text{with} \quad u \neq 0 \tag{3.9}$$

that satisfies the boundary conditions (3.7). Substituting (3.9) into the PDE (3.6), we get:

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda, \quad \lambda \in \mathbb{R} \tag{3.10}$$

where T' is the derivative of T with respect to t and X'' is the second derivative of X with respect to x. So from (3.10), we solve the following ordinary differential equations:

$$T'(t) + \lambda k T(t) = 0 \quad \text{and} \quad X''(x) + \lambda X(x) = 0.$$

To satisfy the boundary conditions (3.7), we must have $X(0) = 0$ and $X(L) = 0$, which leads to the following **Sturm-Liouville** problem:

$$(\mathcal{P}_1) \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \ X(L) = 0, \quad 0 \leq x \leq L. \end{cases}$$

This problem admits eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \text{with} \quad n \in \mathbb{N}^*$$

and corresponding eigenfunctions:

$$X_n(x) = c_1^n \sin\left(\frac{n\pi x}{L}\right), \quad \text{with} \quad n \in \mathbb{N}^* \tag{3.11}$$

where c_1^n are arbitrary constants (see Chapter 1 and 2 for more details on solving $(\mathcal{P}_1)$). Now, for $\lambda_n = n^2\pi^2/L^2$ with $n \in \mathbb{N}^*$, the general solution to the ODE $T'(t) + \lambda_n k T(t) = 0$ is:

$$T_n(t) = c_2^n e^{-k\lambda_n t} \tag{3.12}$$

where c_2^n are arbitrary constants.

Therefore, from (3.11) and (3.12), the general solution of problem (3.6)–(3.7) is:

$$u_n(x,t) = \beta_n e^{-k\lambda_n t} \sin\Big(\frac{n\pi x}{L}\Big), \quad \text{with} \quad n \in \mathbb{N}^*$$

where $\beta_n = c_1^n c_2^n$ and $\lambda_n = n^2\pi^2/L^2$.

Since the problem is linear and homogeneous, then using the superposition principle, we get

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n e^{-k\lambda_n t} \sin\Big(\frac{n\pi x}{L}\Big). \tag{3.13}$$

The use of the initial condition (3.8) in the solution (3.13) gives:

$$f(x) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right), \tag{3.14}$$

which shows that the coefficients β_n are the Fourier coefficients of the function f on the interval $[0, L]$. They are given by:

$$\beta_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right), d\xi, \quad n = 1, 2, \ldots \tag{3.15}$$

Remark 8. *If by substituting (3.15) into (3.13), we can write the solution (3.13) in integral form:*

$$u(x,t) = \int_0^L G\big(x,t;\xi\big) f(\xi) d\xi \tag{3.16}$$

where

$$G\big(x,t;\xi\big) = \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\lambda_n t} \sin\left(\frac{n\pi\xi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \tag{3.17}$$

for all $n \in \mathbb{N}^*$, and G is called the ***Green's function or the heat kernel*** *for the heat equation (3.6) with boundary conditions (3.7).*

It is clear that the function G satisfies the following properties:

(***i***) *G satisfies the heat equation for all x and $t > 0$ provided that $x \neq \xi$;*

(***ii***) *G is continuous at $x = \xi$;*

(***iii***) *G satisfies the boundary conditions (3.7) with respect to the variable x;*

(***iv***) *G is symmetric with respect to x and ξ.*

Example 8. *Determine the solution of the following initial value problem:*

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) = \sin x, \quad 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, \quad t \ge 0. \end{cases} \tag{3.18}$$

Solution: From (3.15), we have

$$\beta_n = \frac{2}{\pi} \int_0^\pi \sin\xi \sin(n\xi) d\xi = \begin{cases} 1 & (n = 1), \\ 0 & (n > 1). \end{cases} \tag{3.19}$$

Therefore, the solution of the problem (3.18) is: $u(x,t) = e^{-t} \sin x$.

3.2.3 Solution of the inhomogeneous problem

The non-homogeneous problem can be expressed as follows:

$$u_t(x,t) = k u_{xx}(x,t) + F(x,t), \quad 0 < x < L, \quad t > 0 \tag{3.20}$$

where F is a source term, with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \ge 0 \tag{3.21}$$

and the initial condition

$$u(x,0) = f(x), \quad x \in [0, L]. \tag{3.22}$$

We look for a solution in the form of a Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{3.23}$$

where the coefficients u_n are functions of t.

Thus, the coefficients u_n are determined by the following formula:

$$u_n(t) = \frac{2}{L} \int_0^L u(x,t) \sin\left(\frac{n\pi x}{L}\right) dx, \quad (n = 1, 2, ...). \tag{3.24}$$

Now, assuming the function u is twice differentiable for $0 \leq x \leq L$ and $t \geq 0$ and that the function F can be represented by the sine Fourier series:

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{3.25}$$

with

$$F_n(t) = \frac{2}{L} \int\limits_0^L F(x,t) \sin\left(\frac{n\pi x}{L}\right) d\xi, \quad (n = 1, 2, ...). \tag{3.26}$$

Then, differentiating (3.24) with respect to t, we get:

$$\begin{aligned} u_n'(t) &= \frac{2}{L} \int\limits_0^L u_t(x,t) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2k}{L} \int\limits_0^L u_{xx}(x,t) \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int\limits_0^L F(x,t) \sin\left(\frac{n\pi x}{L}\right) dx, \end{aligned}$$

and using (3.26), we have

$$u_n'(t) = \frac{2k}{L} \int\limits_0^L u_{xx}(x,t) \sin\left(\frac{n\pi x}{L}\right) dx + F_n(t). \tag{3.27}$$

Using the boundary conditions (3.21), the first term on the right-hand side of (3.27) can be integrated twice by parts, to obtain:

$$\frac{2k}{L} \int\limits_0^L u_{xx}(x,t) \sin\left(\frac{n\pi x}{L}\right) dx = -k\lambda_n u_n(t) \quad \text{with} \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad (n = 1, 2, ...).$$

Hence, from (3.27), we arrive at the following ordinary differential equation:

$$u_n'(t) + k\lambda_n u_n(t) = F_n(t), \tag{3.28}$$

for the functions u_n with $n = 1, 2, ...$ Also, from the initial condition (3.22), we obtain

$$u_n(0) = \frac{2}{L} \int\limits_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = b_n, \quad (n = 1, 2, ...). \tag{3.29}$$

Then, the solution of the initial value problem (3.28)–(3.29) is

$$u_n(t) = \int_0^t e^{-k\lambda_n(t-\tau)} F_n(\tau)d\tau + b_n e^{-k\lambda_n t} \tag{3.30}$$

for all $n = 1, 2, ...$ Finally, by substituting (3.30) into (3.23), we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \int_0^t e^{-k\lambda_n(t-\tau)} F_n(\tau)d\tau \right\} \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t} \sin\left(\frac{n\pi x}{L}\right), \tag{3.31}$$

which is a solution of the problem (3.20)–(3.22).

Example 9. *Determine the solution of the following initial value problem:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = t\sin x, & 0 < x < \pi, \quad t > 0, \\ u(x,0) = 0, & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0. \end{cases} \tag{3.32}$$

Solution: By (3.26), we have

$$F_n(t) = \frac{2t}{\pi} \int_0^{\pi} \sin\xi \sin(n\xi)d\xi = \begin{cases} t, & n = 1, \\ 0, & n > 1. \end{cases} \tag{3.33}$$

Therefore, the solution of problem (3.32) is:

$$u(x,t) = \left(\int_0^t e^{-(t-\tau)} \tau d\tau \right) \sin x = \left(e^{-t} + t - 1\right) \sin x.$$

3.3 Representation of the Solution of the heat equation posed on $\mathbb{R}^n$

In this section, we aim to represent the solution of the heat equation posed on $\mathbb{R}^n$ with $n \ge 1$. First, we give a representation of the solution of the heat equation in $\mathbb{R}$, then we will generalize this result to $\mathbb{R}^n$ for $n > 1$. More precisely, we consider the homogeneous heat equation with an initial condition for all $x \in \mathbb{R}$, and our mathematical model is written as follows:

$$\begin{cases} u_t(x,t) - ku_{xx}(x,t) = 0, & -\infty < x < +\infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < +\infty. \end{cases} \tag{3.34}$$

To find the form of the solution, we will follow here a method based on the Fourier transform, for which we recall briefly some facts.

3.3.1 Review of the Fourier transform

Definition 3. *The Fourier transform F of an integrable function f is defined by:*

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-is\xi} d\xi. \tag{3.35}$$

Then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{isx} ds \tag{3.36}$$

is the inverse Fourier transform of F.

Example 10. *Determine the Fourier transform of the function*

$$f(x) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

Solution: According to (3.35), the Fourier transform is:

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-is\xi} d\xi = \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-is\xi}}{is} \right]_{-a}^{a} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{isa} - e^{-isa}}{is} = \frac{2}{\sqrt{2\pi}\, s} \cdot \frac{e^{isa} - e^{-isa}}{2i} = \sqrt{\frac{2}{\pi}} \frac{\sin(as)}{s}. \end{aligned}$$

Hence, we deduce that:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(as)}{s} e^{isx} ds = \begin{cases} 1, & |x| < a, \\ \frac{1}{2}, & |x| = a, \\ 0, & |x| > a. \end{cases}$$

3.3.2 Solution of the heat equation posed on $\mathbb{R}$

Suppose that the system (3.34) admits a solution u such that u, u_t, u_x, and u_{xx} are absolutely integrable. Applying the Fourier transform to the solution u of the heat equation with respect to the variable x, we obtain:

$$U(s,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,t) e^{-isx} dx, \tag{3.37}$$

and

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} U(s,t) e^{isx} ds. \tag{3.38}$$

Thus, if we can determine the transform (3.37), then the solution of problem (3.34) is given by formula (3.38).

To determine the function U, we differentiate (3.37) with respect to t and using equation (3.34), we get:

$$\begin{aligned} \frac{\partial U}{\partial t}(s,t) &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_t(x,t) e^{-isx} dx \\ &= \frac{k}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_{xx}(x,t) e^{-isx} dx. \end{aligned} \tag{3.39}$$

Integrating by parts twice in (3.39), we have

$$\frac{\partial U}{\partial t}(s,t) = \frac{k}{\sqrt{2\pi}} \Big(u_x(x,t) e^{-isx} + isu(x,t) e^{-isx} \Big) \Big|_{-\infty}^{\infty} - \frac{ks^2}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,t) e^{-isx} dx. \tag{3.40}$$

Now, if we assume that u and u_x vanish as $|x| \to \infty$, then equation (3.40) reduces to:

$$\frac{\partial U}{\partial t}(s,t) = -\frac{ks^2}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,t) e^{-isx} dx = -ks^2 U(s,t).$$

Also, from the initial condition $u(x,0) = f(x)$ and (3.37), we have

$$U(s,0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x,0) e^{-isx} dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-isx} dx = F(s).$$

Finally, using the Fourier transform, problem (3.34) is equivalent to:

$$\begin{cases} \frac{\partial U}{\partial t}(s,t) = -ks^2 U(s,t), \\ U(s,0) = F(s), \end{cases} \tag{3.41}$$

The solution of system (3.41) is given by:

$$U(s,t) = F(s)e^{-ks^2 t}. \tag{3.42}$$

Consequently, from (3.38) and (3.42), we deduce that:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \exp\big(-ks^2 t + isx\big) ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\big(is(x-\xi) - ks^2 t\big) f(\xi) d\xi ds. \end{aligned} \tag{3.43}$$

Using the **Fubini theorem**, we can interchange the order of integration in (3.43) to write:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi; t) f(\xi) d\xi, \tag{3.44}$$

where

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\big(isx - ks^2 t\big) ds. \tag{3.45}$$

Also, by Euler's formula

$$e^{ix} = \cos x + i \sin x,$$

we can compute the integral (3.45) as follows:

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ks^2 t} \big(\cos(sx) + i \sin(sx)\big) ds.$$

Since $e^{-ks^2 t}\cos(sx)$ is an even function of s and $e^{-ks^2 t}\sin(sx)$ is odd, we have

$$\int_{-\infty}^{\infty} e^{-ks^2 t} \sin(sx) ds = 0,$$

and

$$G(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ks^2t}\cos(sx)ds=\frac{1}{\pi}\int_{0}^{\infty}e^{-ks^2t}\cos(sx)ds. \tag{3.46}$$

If we set $z=s\sqrt{kt}$ in (3.46), then

$$G(x,t)=\frac{1}{\sqrt{kt}\,\pi}\int_{0}^{\infty}e^{-z^2}\cos\left(\frac{xz}{\sqrt{kt}}\right)dz.$$

But

$$\int_{0}^{\infty}e^{-z^2}\cos\left(\frac{xz}{\sqrt{kt}}\right)dz=\frac{\sqrt{\pi}}{2}e^{-x^2/4kt},$$

so

$$G(x,t)=\frac{1}{\pi}\int_{0}^{\infty}e^{-ks^2t}\cos(sx)ds=\frac{1}{2\sqrt{\pi kt}}e^{-x^2/4kt}. \tag{3.47}$$

Finally, substituting (3.47) into (3.44) gives a **representation of the solution** u in the form

$$u(x,t)=\frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty}\exp\left(\frac{-(x-\xi)^2}{4kt}\right)f(\xi)d\xi, \tag{3.48}$$

for the initial value problem (3.34).

Remark 9. *The function*

$$G(x-\xi,t)=\frac{1}{2\sqrt{\pi kt}}\exp\left(-\frac{(x-\xi)^2}{4kt}\right),\quad t>0 \tag{3.49}$$

is called a ***fundamental solution*** *or* ***Green's function*** *for the heat equation in the infinite domain* $-\infty<x<+\infty$.

It is clear that the function G *satisfies the following properties:*

- ***(i)*** G *admits continuous partial derivatives with respect to* x *and* t *for all* $-\infty<x<+\infty$ *and* $t>0$, *and satisfies the heat equation for all* x *and* $t>0$ *provided* $x\neq\xi$;
- ***(ii)*** G *is continuous at* $x=\xi$;
- ***(iii)*** G *vanishes as* $|x|\to\infty$ *for all* $t>0$;
- ***(iv)*** G *is symmetric with respect to* x *and* ξ;

(v) *G satisfies*

$$\int_{-\infty}^{\infty} G(x-\xi,t)\,d\xi = 1.$$

Example 11. *Solve the following problem:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, & -\infty < x < +\infty, \quad t > 0, \\ u(x,0) = e^{-x}, & -\infty < x < +\infty. \end{cases} \tag{3.50}$$

Solution: According to formula (3.48), the solution of problem (3.50) is written as:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4t} - \xi\right) d\xi.$$

We have:

$$\begin{aligned} \frac{(x-\xi)^2}{4t} + \xi &= \frac{1}{4t}\left(x^2 - 2x\xi + \xi^2 + 4t\xi\right) \\ &= \frac{1}{4t}\left(x^2 + \xi^2 + 4t^2 - 2\xi x + 4\xi t - 4xt + 4xt - 4t^2\right) \\ &= \frac{(\xi + 2t - x)^2}{4t} + x - t. \end{aligned}$$

Now, we make the following change of variable: $z = \dfrac{\xi + 2t - x}{2\sqrt{t}}$, hence

$$u(x,t) = \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \cdot 2\sqrt{t}\,dz = \frac{2\sqrt{t}\,e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-z^2} dz = e^{t-x},$$

because $\displaystyle\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$.

3.3.3 Solution and regularity of the heat equation posed on $\mathbb{R}^n$

Using the previous result, we can provide an explicit representation of the solution to the Cauchy problem for the heat equation in dimension $n \geq 1$. More precisely, by applying the Fourier transform, we prove the following result:

Theorem 8. *Let $\varphi \in C(\mathbb{R}^n)$ and consider the Cauchy problem:*

$$\begin{cases} \frac{\partial u}{\partial t}(\mathbf{x},t) - k\Delta_{\mathbf{x}} u(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^n, \quad t \in [0,T], \\ u(\mathbf{x},0) = \varphi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^n, \end{cases} \tag{3.51}$$

where $\Delta_{\mathbf{x}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ *denotes the spatial Laplacian, and* $T > 0$.

Then, the system (3.51) *admits a unique solution*

$$u \in C^{\infty}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$$

given by the explicit formula:

$$u(\mathbf{x},t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|\mathbf{x}-\xi|^2}{4kt}\right) \varphi(\xi)\, d\xi. \tag{3.52}$$

Remark 10. *The formula* (3.52) *is also known as the* ***Poisson formula*** *for the heat equation.*

3.4 Special equations

3.4.1 Bernoulli equation

A Bernoulli differential equation is a first-order differential equation of the form:

$$y' = a(t)y + b(t)y^m, \tag{3.53}$$

where m is different from 0 and 1, and a and b are functions defined on an open interval $I \subset \mathbb{R}$ with real values.

In general, m is a natural integer, but m can be taken as a real number provided that we seek positive values of y. Usually, a and b are continuous functions.

Solving Method: If $m \neq 0$ and $m \neq 1$, the idea of the method is to divide both sides of (3.53) by y^m, obtaining:

$$\frac{y'}{y^m} - a(t)\frac{1}{y^{m-1}} = b(t). \tag{3.54}$$

We perform the change of variable by setting $z = \dfrac{1}{y^{m-1}}$, which gives:

$$\frac{1}{m-1} z' - a(t)z = b(t). \tag{3.55}$$

Thus, z satisfies a linear first-order differential equation. We solve it and deduce an expression for y.

Example 12. *Determine the solution of the following Bernoulli equation:*

$$y' = y + t^2 y^2. \tag{3.56}$$

Solution: Divide equation (3.56) by y^2, which yields:

$$\frac{y'}{y^2} - \frac{1}{y} = t^2.$$

By setting $z = \frac{1}{y}$, we obtain a linear equation

$$z' + z = -t^2.$$

The solution of the associated homogeneous equation is $z_h = \lambda e^{-t}$ with $\lambda \in \mathbb{R}$. By the method of variation of constants, we find

$$z(t) = \lambda e^{-t} - t^2 + 2t - 2, \quad \lambda \in \mathbb{R}.$$

Finally, we get

$$y(t) = \frac{1}{\lambda e^{-t} - t^2 + 2t - 2}, \quad \lambda \in \mathbb{R}.$$

3.4.2 Riccati equation

A Riccati differential equation is a first-order differential equation of the form:

$$y' = a(t)y^2 + b(t)y + c(t), \tag{3.57}$$

where a, b, and c are three functions, often chosen continuous on a common interval $I \subset \mathbb{R}$ with real values.

The integration of a Riccati differential equation requires the knowledge of a particular solution of this equation.

Solving Method: We assume a particular solution y_1 and set $y = z + y_1$. Replacing y by this value in (3.57) we get:

$$z' = a(t)z^2 + \big(2a(t)y_1 + b(t)\big)z,$$

where a, b, and c are continuous functions on I. Thus, z is a solution of a Bernoulli equation, which has already been treated above.

Example 13. *Solve the following differential equation:*

$$t(t-1)y' + y^2 - (2t+1)y = -2t. \tag{3.58}$$

Solution: Equation (3.58) is a Riccati equation with:

$$a(t) = \frac{-1}{t(t-1)}, \quad b(t) = \frac{2t+1}{t(t-1)}, \quad \text{and} \quad c(t) = \frac{2}{1-t}.$$

Note that $y_1 = t$ is a particular solution of equation (3.58). Setting $y = t + z$ gives:

$$t(t-1)z' - z + z^2 = 0.$$

Dividing this equation by $t(t-1)$ yields a Bernoulli equation with $m = 2$:

$$z' = \frac{1}{t(t-1)}z - \frac{1}{t(t-1)}z^2,$$

from which it follows that $z(t) = \dfrac{t-1}{\lambda t - 1}$ with $\lambda \in \mathbb{R}$, and finally:

$$y(t) = t + \frac{t-1}{\lambda t - 1}.$$

3.4.3 Clairaut equation

Lagrange equations are equations that can be written in the form:

$$y = a\big(y'\big)\, t + b\big(y'\big) \tag{3.59}$$

where a and b are differentiable functions on an interval $I \subset \mathbb{R}$.

The Clairaut equation is a particular case of the Lagrange equation with $a\big(y'\big) = y'$, that is, it takes the form:

$$y = ty' + b\big(y'\big). \tag{3.60}$$

To solve this type of equation, we set $y' = s$ or $\dfrac{dy}{dt} = s$ and look for an expression for $\dfrac{dy}{dt}$. We have

$$y = a\big(y'\big)t + b\big(y'\big) = a(s)t + b(s) \quad \text{so} \quad \frac{dy}{dt} = a(s) + t\frac{d}{dt}a(s) + \frac{d}{dt}b(s),$$

hence

$$\frac{dy}{dt} = a(s) + t\frac{d}{ds}a(s)\frac{ds}{dt} + \frac{d}{ds}b(s)\frac{ds}{dt} = a(s) + ta'(s)\frac{ds}{dt} + b'(s)\frac{ds}{dt}.$$

Since $\frac{dy}{dt} = s$, then

$$a(s) + ta'(s)\frac{ds}{dt} + b'(s)\frac{ds}{dt} = s,$$

which is a differential equation for the unknown function $t(s)$. Once $t(s)$ is found, we get $y(t) = a(s)t(s) + b(s)$.

Example 14. *Solve the following Clairaut equation:*

$$y = ty' + \left(y'\right)^2 + 1. \tag{3.61}$$

Solution: We set $y' = s$ or $\frac{dy}{dt} = s$, then

$$y = ts + s^2 + 1 \quad \text{and} \quad \frac{dy}{dt} = s + t\frac{ds}{dt} + 2s\frac{ds}{dt}.$$

Since $\frac{dy}{dt} = s$, then

$$s + t\frac{ds}{dt} + 2s\frac{ds}{dt} = s, \quad \text{hence} \quad (t + 2s)\frac{ds}{dt} = 0.$$

So we have either $t + 2s = 0$ or $\frac{ds}{dt} = 0$.

If $t + 2s = 0$, then $(t, y) = (-2s, -s^2 + 1)$, hence $y = -\frac{t^2}{4} + 1$.

If $\frac{ds}{dt} = 0$, then $s = \text{constant} = \lambda$, and thus $y = \lambda t + \lambda^2 + 1$.

3.5 Exercises

***Exercise* 3.1.** *Consider the following problem:*

$$(\mathcal{H}) \begin{cases} u_{xx}(x,t) - u_t(x,t) = e^{-x} & 0 < x < L;\ t > 0, \\ u(x,0) = e^{-x} - 1 & 0 \le x \le L, \\ u(0,t) = u(L,t) = 0 & t > 0. \end{cases}$$

1) *Determine a function* $u_p(x)$ *solution of* $(\mathcal{H})$ *independent of* t.

2) *Define* $v(x,t) = u(x,t) - u_p(x)$. *Show that* v *is a solution of the following problem:*

$$(\mathcal{H}_p)\begin{cases} v_{xx}(x,t) = v_t(x,t) & 0 < x < L;\ t > 0, \\ v(x,0) = \left(e^{-L} - 1\right)\frac{x}{L} & 0 \leq x \leq L, \\ v(0,t) = v(L,t) = 0 & t > 0. \end{cases}$$

3) *Using the* ***method of separation of variables****, solve the problem* $(\mathcal{H}_p)$ *and deduce the solution* u *of* $(\mathcal{H})$.

Exercise 3.2. 1) *Using the* ***method of separation of variables****,*
determine the solution of the following problem:

$$(\mathcal{P})\begin{cases} u_t(x,t) - ku_{xx}(x,t) = 0 & 0 < x < L;\ t > 0, \\ u(x,0) = f(x) & 0 \leq x \leq L, \\ u_x(0,t) = u_x(L,t) = 0 & t \geq 0. \end{cases}$$

where f *is a given function.*

2) *Determine the Green's function* G *for the problem* $(\mathcal{P})$.

3) *For* $f(x) = x$ *and* $L = \pi$, *solve the problem* $(\mathcal{P})$.

Exercise 3.3. *Consider the following problem:*

$$(\mathcal{A})\begin{cases} u_t(x,t) - k\,u_{xx}(x,t) = 0 & 0 < x < L;\ t > 0, \\ u(x,0) = f(x) & 0 \leq x \leq L, \\ u(0,t) = T_0,\ u(L,t) = T_1 & t \geq 0. \end{cases}$$

where k, T_0 *and* T_1 *are constants and* f *is a continuous function on* $[0,L]$.

1) *Determine a function* $u_p(x)$ *solution of* $(\mathcal{A})$ *independent of* t.

2) *Define* $u(x,t) = v(x,t) + u_p(x)$. *Show that* v *is a solution of the following problem:*

$$(\mathcal{B}_p)\begin{cases} v_t(x,t) = k\,v_{xx}(x,t) & 0 < x < L;\ t > 0, \\ v(x,0) = f(x) - u_p(x) & 0 \leq x \leq L, \\ v(0,t) = v(L,t) = 0 & t \geq 0. \end{cases}$$

3) *Using* ***the method of separation of variables****, solve the problem* $(\mathcal{B}_p)$ *and deduce the solution* u *of* $(\mathcal{A})$.

***Exercise* 3.4.** *Consider the following problem:*

$$(\mathcal{P})\begin{cases} u_t(x,t)-k\,u_{xx}(x,t)=h(x,t) & -\infty<x<+\infty;\ t>0,\\ u(x,0)=0 & -\infty<x<+\infty \end{cases}$$

*where k is a constant. Using **the Fourier transform**, show that the solution of the problem $(\mathcal{P})$ is:*

$$u(x,t)=\int\limits_0^t\int\limits_{-\infty}^{\infty}G\big(x-\xi,t-\tau\big)h(\xi,\tau)d\xi d\tau$$

with

$$G\big(x-\xi,t-\tau\big)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\exp\big[is(x-\xi)-ks^2(t-\tau)\big]ds$$

being the Green's function for the problem $(\mathcal{P})$.

***Exercise* 3.5.** *Consider the following problem:*

$$(\mathcal{T})\begin{cases} u_t(x,t)-k\,u_{xx}(x,t)=0 & x>0;\ t>0,\\ u(x,0)=f(x) & x\geq 0,\\ u(0,t)=0 & t\geq 0 \end{cases}$$

where k is a constant and f is a given function. Suppose that the problem $(\mathcal{T})$ admits a solution u such that u, u_t, u_x, and u_{xx} are absolutely integrable.

To solve the problem $(\mathcal{T})$, define the Fourier transform of u with respect to the variable x by:

$$U(s,t)=\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}u(x,t)\sin(sx)dx \quad \text{with} \quad u(x,t)=\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}U(s,t)\sin(sx)ds. \tag{3.62}$$

1)- Using (3.62), show that the problem $(\mathcal{T})$ is equivalent to:

$$(\mathcal{T}')\begin{cases} \frac{\partial U}{\partial t}(s,t)+ks^2U(s,t)=0,\\ U(s,0)=F(s), \end{cases}$$

where

$$F(s)=\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}f(x)\sin(sx)dx.$$

2)- Solve the problem $(\mathcal{T}')$ *and deduce that*

$$u(x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(\xi)\sin(s\xi)\sin(sx)e^{-ks^2t}\ d\xi ds$$

is a solution of $(\mathcal{T})$.

***Exercise* 3.6.** *Consider the following problem:*

$$(\mathcal{D})\ \begin{cases} u_t(x,t) - k\,u_{xx}(x,t) = 0 & x > 0;\ t > 0, \\ u(x,0) = 0 & x \geq 0, \\ u_x(0,t) = h(t) & t \geq 0 \end{cases}$$

where k *is a constant and* h *is a given function. Suppose that the problem* $(\mathcal{D})$ *admits a solution* u *such that* u, u_t, u_x, *and* u_{xx} *are absolutely integrable.*

1)- Show that the problem $(\mathcal{D})$ *is equivalent to:*

$$(\mathcal{D}')\ \begin{cases} \frac{\partial U}{\partial t}(s,t) + ks^2U(s,t) = -k\sqrt{\frac{2}{\pi}}h(t), \\ U(s,0) = 0, \end{cases}$$

if

$$U(s,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos(sx)dx \quad \textit{with} \quad u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty U(s,t)\cos(sx)ds. \tag{3.63}$$

2)- Solve the problem $(\mathcal{D}')$ *and deduce that*

$$u(x,t) = -\sqrt{\frac{k}{\pi}}\int_0^t \frac{\exp\left(\frac{-x^2}{4k(t-\tau)}\right)}{\sqrt{t-\tau}}h(\tau)\ d\tau$$

is a solution of $(\mathcal{D})$.

3)- For $h(t) \equiv 1$ *and* $k = 1$, *determine the solution of the problem* $(\mathcal{D})$.

Exercise* 3.7.** $\big($The heat equation in 2D***$\big)$. *Consider the following problem:*

$$u(x,y,t) = c^2\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big), \quad 0 < x < a,\ 0 < y < b,\ t > 0,\ c > 0, \tag{3.64}$$

with the boundary conditions

$$\begin{cases} u(x,0,t)=0, \quad u(x,b,t)=0, \quad 0<x<a,\ \ t>0, \\ u(0,y,t)=0, \quad u(a,y,t)=0, \quad 0<y<b,\ \ t>0 \end{cases} \tag{3.65}$$

and the initial condition

$$u(x,y,0)=f(x,y), \quad 0<x<a,\ \ 0<y<b. \tag{3.66}$$

where f is a given function.

1)- Using the method of separation of variables, determine the solution of the problem (3.64)–(3.65).

2)- For $f(x,y)=xy$, $a=b=\pi$ and $c=1$, solve the problem (3.64)–(3.65).

***Exercise* 3.8.** *1) Solve for x in $\mathbb{R}-\{0\}$ the differential equation*

$$\frac{dy}{dx}-\frac{y}{x}=xy^2.$$

2) Solve for $x\neq 0$ the following differential equation

$$x\frac{dy}{dx}+6y=3xy^{\frac{4}{3}}.$$

3) Solve the Riccati equation

$$\frac{dy}{dx}=(y-x)^2+1.$$

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