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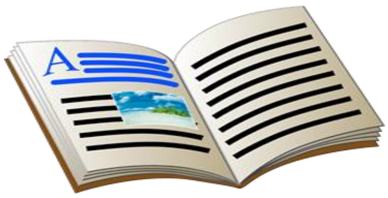

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Differential Geometry

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Chapter 1

Introduction

These lecture notes are intended for third-year undergraduate students (L3) in Mathematics, second semester. They provide an introduction to the fundamental concepts of differential geometry, focusing on the analytical tools that allow us to describe functions and subsets of normed vector spaces locally.

The course begins with a thorough study of the Local Inversion Theorem and the Implicit Function Theorem, two central results of differential calculus. These theorems are then generalized by the Constant Rank Theorem, which offers a unified view of immersions, submersions, and maps of constant rank. The final part of the notes is devoted to the notion of submanifolds, approached both through equations and regular values.

Each chapter is illustrated with numerous exercises, ranging from simple applications to more elaborate problems, in order to prepare the student for concrete geometric situations. Detailed examples are also provided to facilitate understanding of abstract concepts.

The necessary prerequisites correspond to knowledge acquired in previous semesters: basic differential calculus (partial derivatives, differential, C^k functions), normed vector spaces, compactness, connectedness, and elementary topology. Familiarity with Banach spaces is an asset, but essential reminders are given in the text.

The aim of this course is to provide students with the fundamental tools needed to approach more advanced fields such as Riemannian geometry, Lie groups, or partial differential equations. We hope that these notes, enriched by tutorials and practical sessions, will spark interest in the beauty and power of differential geometry.

Contents of the notes:

1. Local Inversion Theorem and Implicit Function Theorem.
2. Constant Rank Theorem: immersions, submersions.
3. Submanifolds: definition, examples, characterization by regular values.

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Good reading and good work!

H. Banouh
Academic year 2025/2026

Chapter 2

Local Inversion Theorem

§2.1 Differential

Definition 2.1.1

Let $f : E \to F$ and $a \in E$. We say that f is differentiable at a *iff* $\exists L \in \mathcal{L}(E, F)$ such that $\forall h \in E$

$$\begin{aligned} & f(a+h) - f(a) = L \bullet h + o(\|h\|) \\ \iff & f(a+h) - f(a) = L \bullet h + \|h\|\epsilon(h) \end{aligned} \tag{2.1}$$

with $\lim_{h\to 0} \epsilon(h) = 0$. Or equivalently

$$\lim_{h\to 0_E} \frac{\|f(a+h) - f(a) - L \bullet h\|_F}{\|h\|_E} = 0. \tag{2.2}$$

We say that f is differentiable on an open set $U \subset E$ if it is differentiable at every point a in U.

Remark 2.1. *We will denote $L = d_a f$ in what follows and call it the differential of f at a.*

Proposition 2.1.1

If f is differentiable at a then the differential is unique.

Proof. See problem set. ∎

Proposition 2.1.2

If the function $a \longmapsto d_a f$ is continuous at 0_E then f is continuous at a.

Proof. By definition 2.1 and setting $h = x - a$ we obtain

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{h\to 0} f(a+h)\\ &= \lim_{h\to 0}\left[f(a) + df_a \bullet h + \|h\|\epsilon(h)\right]\\ &= f(a) + \lim_{h\to 0}\left[df_a \bullet h\right] + \lim_{h\to 0}\left[\|h\|\epsilon(h)\right]\\ &= f(a) + \lim_{h\to 0}\left[df_a \bullet h\right] + \lim_{h\to 0}\left[\|h\|\epsilon(h)\right]\end{aligned}$$

and since $d_a f \in \mathcal{L}(E, F)$ then $\lim_{h\to 0}\left[df_a \bullet h\right] = 0$ so

$$\lim_{x\to a} f(x) = f(a).$$

■

Exercise 2.1. *Let*

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}\\ (x, y) &\mapsto f(x, y) = x\sqrt{y^2+2} + 1\end{aligned}$$

- Compute $d_{(0,0)} f$.

Solution. *In this case* $\begin{cases} h &= (h_1, h_2)\\ a &= (0,0)\end{cases}$. *By the definition of the differential at a point*

$$f\left((0,0) + (h_1, h_2)\right) = f\left(0,0\right) + d_{(0,0)} f \bullet (h_1, h_2) + \|h\|\epsilon(h).$$

We have $f\left(0,0\right) = 1$ *so*

$$\begin{aligned} f\left(h_1, h_2\right) &= 1 + h_1\sqrt{h_2^2 + 2}\\ &= f\left(0,0\right) + h_1\sqrt{2\left(1 + \frac{h_2^2}{2}\right)}\\ &= f\left(0,0\right) + \sqrt{2}h_1\sqrt{1 + \frac{h_2^2}{2}}\end{aligned}$$

Writing the limited expansion of the function $h_2 \longmapsto \sqrt{1 + \frac{h_2^2}{2}}$ *to order* 3 *near* 0 *we obtain*

$$\begin{aligned} f\left(h_1, h_2\right) &= f\left(0,0\right) + \sqrt{2}h_1\left(1 + \frac{h_2^2}{4} + o(h_2^2)\right)\\ &= f\left(0,0\right) + \sqrt{2}h_1 + \frac{\sqrt{2}}{4}h_1 h_2^2 + \sqrt{2}h_1 o(h_2^2)\end{aligned}$$

and since $\lim_{(h_1,h_2)\to(0,0)} \frac{\sqrt{2}}{4}h_1 h_2^2 + \sqrt{2}h_1 o(h_2^2) = 0$ *we write*

$$f\left(h_1, h_2\right) = f\left(0,0\right) + \sqrt{2}h_1 + o(\|h\|)$$

So we choose $d_{(0,0)} f \bullet (h_1, h_2) = \begin{pmatrix}\sqrt{2}\\ 0\end{pmatrix}(h_1, h_2) = \sqrt{2}h_1.$

Proposition 2.1.3

Let E, F and G be normed vector spaces and let $f : E \to F$ and $g : F \to G$ be two functions differentiable at a and $b = f(a)$ respectively then

- $$d_a(g \circ f) = d_{f(a)}g \circ d_a f.eq : diffcomp \tag{2.3}$$
- if in addition $f(a) \neq 0$ then
$$d_a\left(\frac{g}{f}\right) = \frac{f(a)d_a g - d_a f g(a)}{[f(a)]^2}$$
- $$d_a(\frac{1}{f}) = \frac{-d_a f}{[f(a)]^2}$$

§2.2 Higher Order Differentials

Let $U \subset E$ be an open subset, $f : U \to F$ a differentiable map on U and $a \mapsto d_a f$ its differential. If the function $a \mapsto d_a f$ is also differentiable at a then we call its differential the second-order differential or second derivative of f and we denote $d_a(d_a f) = d_a^2 f$ and we have

$$df(a+h) = d_a f + d_a^2 f \bullet h + \|h\|\epsilon(h).$$

Similarly, we can define the k-th order differential ($k \geq 2$) of f at a as a $k-$linear map denoted

$$d_a^k f = d_a(d_a^{k-1} f).$$

Definition 2.2.1

We say that a map $f : U \subset E \to F$ is of class $\mathcal{C}^1$ on U if it is differentiable on U and its differential df is continuous on U. Similarly, we say that a map is of class $\mathcal{C}^k$ on U iff it is k times differentiable and $d^k f$ is also continuous.

Definition 2.2.2

Let $a \in U \subset E$ and $f : U \to F$. If the partial derivatives $\frac{\partial f}{\partial x_i}$ exist at a and are continuous then $f \in \mathcal{C}^1(U, F)$. If the partial derivatives themselves possess partial derivatives then we call them second partial derivatives of f and we denote

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}(a)\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

If these second partial derivatives exist in a neighborhood of point a and are continuous we say that f is of class $\mathcal{C}^2$ or $f \in \mathcal{C}^2(U, F)$. We define by recurrence the $k-$th order partial derivatives and the notion of functions of class $\mathcal{C}^k$. Finally, we say that a function is of class $\mathcal{C}^\infty$ at a if all its partial derivatives of any order exist in a neighborhood of a and they are all continuous at a.

§2.3 Mean Value Theorem and Inequality:

Lemma 2.3.1

Let $[a, b]$ be a closed interval of $\mathbb{R}$, F a normed vector space and let $f : [a, b] \to F$ and $g : [a, b] \to \mathbb{R}$ be two functions continuous on $[a, b]$ and differentiable on $]a, b[$ such that:

$$\forall t \in \,]a, b[$$

Theorem 2.3.1: Mean value theorem

et U be an open subset of a normed vector space E of dimension n and let $f : U \to \mathbb{R}$. Choose a point $a \in U$ and a vector $h \in E$ such that the set (called closed segment)

$$[a, a + h] := \{a + th, t \in [0, 1]\} \subset U.$$

Assume that f is differentiable at every point of U. Then $\exists \theta \in \,]0, 1[$ such that

$$f(a + h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a + \theta h).$$

Proof. For $t \in [0, 1]$ set $\varphi(t) = f(a + th)$. The function φ is clearly differentiable on $]0, 1[$ and we have

$$\begin{aligned} \varphi'(t) &= d_{a+th} f \bullet h \\ &= J_{a+th} f \bullet h. \end{aligned}$$

Applying the mean value theorem to the function φ : $\exists\theta \in]0,1[$ such that

$$\begin{aligned}\varphi(1)-\varphi(0) &= \varphi'(\theta)\\ &= J_{a+\theta h} f \bullet h\\ &= f(a+h)-f(a)\\ &= \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a+\theta h).\end{aligned}$$

■

Remark 2.2. *The previous result is not valid if the function f is vector-valued. But we can have an inequality.*

Theorem 2.3.2

Let $f : U \subset E \to F$ and let $a \in U$ and $h \in E$ such that the closed segment $[a, a+h] \in U$. Assume f is differentiable on the open segment

$$]a, a+h[:= \{a+th, t \in]0,1[\}$$

and that $\exists M > 0$ such that $\forall x \in]a, a+h[$:

$$\|d_x f\| \leq M$$

then

$$\|f(a+h)-f(a)\|_F \leq M\|h\|_E.$$

Proof. Recall the notion of norm of a linear map $A \in \mathcal{L}(E,F)$

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{\|x\|_E=1} \|A \bullet x\|_F = \sup_{\|x\|_E\neq 0} \frac{\|A \bullet x\|}{\|x\|_E}.$$

Assume that $\exists M > 0, \forall a \in U : \|d_a f\| \leq M$. For $\epsilon > 0$ define the set

$$A_\epsilon = \{t \in [0,1] : \|f(a+th)-f(a)\|_F \leq (M+\epsilon)t\|h\|_E\}$$

and the function $t \mapsto g_\epsilon(t) = (M+\epsilon)t\|h\|_E - \|f(a+th)-f(a)\|_F$. Since the function g_ϵ is continuous on $[0,1]$ and the set A_ϵ is closed bounded then it is bounded above. Let $\sup_{t\in[0,1]} A_\epsilon = t_0$. We will show that $t_0 = 1$. By contradiction: assume that $t_0 < 1$ then $\forall t \in [t_0, 1]$ we have

$$\|f(a+th)-f(a)\|_F > (M+\epsilon)t\|h\|_E.$$

By definition $t_0 \in A_\epsilon$ so

$$\|f(a+t_0h)-f(a)\|_F \leq (M+\epsilon)t_0\|h\|_E.$$

Subtracting the second inequality from the first we obtain

$$\|f(a+th)-f(a)\|_F - \|f(a+t_0h)-f(a)\|_F \geq (M+\epsilon)\,(t-t_0)\,\|h\|_E$$

and applying the triangle inequality

$$\|f(a+th)-f(a+t_0h)\|_F \geq (M+\epsilon)\,(t-t_0)\,\|h\|_E.$$

Since f is differentiable at point $a+t_0h$ and its differential gives

$$d_{a+t_0h}f \bullet h := \lim_{t\to t_0}\frac{f(a+th)-f(a+t_0h)}{t-t_0}$$

and taking norms, we have

$$\|d_{a+t_0h}f\| \geq M+\epsilon > M$$

which contradicts the fact that $\|d_xf\| \leq M, \forall x \in E$ so we must have $t_0 = \sup_{t\in[0,1]} A_\epsilon = 1$. Taking the limit $\epsilon \to 0$ and $t = 1$ we get

$$\|f(a+th)-f(a)\|_F \leq M\|h\|_E$$

■

Definition 2.3.1: Lipschitz function

A function $f : U \subset E \to F$ is

- $K-$Lipschitz if $\forall x, y \in U$
$$\|f(y)-f(x)\|_F \leq K\|y-x\|_E.$$
- If the constant $0 < K < 1$, we say the function f is contracting.
- The function f is locally Lipschitz if $\forall a \in U$ there exists an open neighborhood of a on which f is Lipschitz.

Lemma 2.3.2

Let $f : [a,b] \to F$ and $g : [a,b] \to \mathbb{R}$ be two functions continuous on their domain. Assume that f and g admit right derivatives at every point $t \in \,]a,b[$ such that

$$\|f'_d(t)\|_F \leq g'_d(t), \forall t \in \,]a,b[$$

then

$$\|f(b)-f(a)\|_F \leq |g(b)-g(a)|\,.$$

Theorem 2.3.3

Let E and F be two Banach spaces and let $f : U \subset E \to F$ where U is a convex open set and let $K > 0$ such that $\forall x \in U; \|d_x f\|_F \leq K$ then f is $K-$Lipschitz.

Proof. For $t \in [0, 1]$ set

$$\begin{aligned} g(t) &= x + t(y - x) \\ h(t) &= f \circ g(t). \end{aligned}$$

We have $g'(t) = y - x$ and

$$\begin{aligned} h'(t) &= d_{g(t)} f \bullet g'(t) \\ &= d_{g(t)} f \bullet (y - x) \end{aligned}$$

so

$$\begin{aligned} \|h'(t)\|_F &\leq \|d_{g(t)} f\|_F \|y - x\|_E \\ &\leq K\|y - x\|_E. \end{aligned}$$

Applying lemma 2.3.2 to functions g and h on the interval $[0, 1]$ we find that

$$\begin{aligned} \|h(1) - h(0)\|_F &= \|f(y) - f(x)\|_F \\ &\leq K\|y - x\|_E. \end{aligned}$$

■

The mean value theorem has numerous applications, including the characterization of functions with zero differential on connected open sets.

Definition 2.3.2

A subset of a topological space is connected if it admits no subset that is both open and closed other than the empty set and itself.

Theorem 2.3.4

Let $f : U \subset E \to F$ be a differentiable function on a connected open set U, such that $d_a f \equiv 0$ for all $a \in U$ then f is constant.

Proof. For every $a \in U$, there exists $r > 0$ such that the open ball $B(x, r) \subset U$. This ball being convex and since $df \equiv 0$, the mean value theorem shows that $f(y) = f(x)$ for every $y \in B(x, r)$ and therefore f is locally constant. Fix $b \in U$. The set $f^{-1}(\{f(b)\}) \subset U$ is non-empty since it contains b, and closed by continuity of f (the singleton $\{f(b)\}$ being closed). From the above, this set is also open. Since U is connected, we therefore have $f^{-1}(\{f(x)\}) = U$. In other words, $f(b) = f(a)$ for all $b \in U$. ■

Theorem 2.3.5

Let E and F be two Banach spaces and U a convex open subset of E. Consider a sequence of differentiable functions on E denoted $\{f_n\}_{n\in\mathbb{N}}$ such that

1. $\exists a \in E$ such that the sequence $\{f_n(a)\}_{n\in\mathbb{N}}$ is Cauchy in F.
2. The sequence of functions $\{df_n\}_{n\in\mathbb{N}}$ with $df_n : U \to \mathcal{L}(E, F)$ converges uniformly to $g : U \to \mathcal{L}(E, F)$.

Then

1. The sequence $\{f_n\}_{n\in\mathbb{N}} \overset{C.U}{\to} f$ on each bounded subset of U.
2. The limit g is differentiable and $df = g$.

§2.4 Implicit Function Theorem.

§2.4.1 Diffeomorphism

Definition 2.4.1

Let E and F be two Banach spaces and $U \subset E, V \subset F$ two open sets. We say that $f : U \to V$ is a diffeomorphism of class $\mathcal{C}^1$ or $\mathcal{C}^1-$diffeomorphism iff

1. f is bijective.
2. f and its inverse f^{-1} are of class $\mathcal{C}^1$.

Exercise 2.2. *Show that the function $\varphi : (u, v) \mapsto (u+v, uv)$ is a $\mathcal{C}^1-$diffeomorphism from $U = \{(u, v) \in \mathbb{R}^2; u > v\}$ onto an open set to be determined.*

Solution. *For $u > v$ set*

$$(s,t) = \varphi(u,v) = \begin{cases} s &= u+v \\ t &= uv \end{cases}.$$

Recall (u,v) are the solutions of the quadratic equation $x^2 - sx + t = 0$ and therefore the discriminant Δ must be strictly positive:

$$\Delta = s^2 - 4t > 0 \implies \begin{cases} u &= \frac{s+\sqrt{s^2-4t}}{2} \\ v &= \frac{s-\sqrt{s^2-4t}}{2} \end{cases}.$$

In this case, φ is a bijection from U to $V = \{(s,t) \in \mathbb{R}^2; s^2 > 4t\}$ (see 2.1) and

$$\varphi^{-1}(s,t) = (u,v) = \left(\frac{s+\sqrt{s^2-4t}}{2}, \frac{s-\sqrt{s^2-4t}}{2}\right)$$

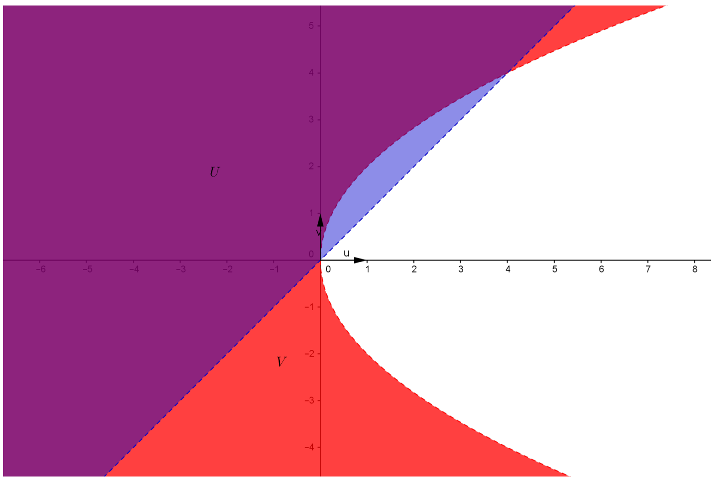


Figure 2.1: U and V

Lemma 2.4.1

If $u \in \mathcal{L}(E,F)$ such that $\|u\|_{\mathcal{L}(E,F)} < 1$ then $Id_F - u \in Isom(E,F)$.

Proof. We know that $\|u\|_{\mathcal{L}(E,F)} := \sup_{x\in B(0,1)} \|u \bullet x\|_F < 1$ so the series $\sum_{n=0}^{\infty} u^n$ is normally convergent and $\lim_{n\to\infty} u^n = \lim_{n\to\infty} u^{n+1} = 0_{\mathcal{L}(E,F)}$. We can write

$$\begin{aligned}(Id_F - u)\left(\sum_{i=0}^{n} u^i\right) &= \left(\sum_{i=0}^{n} u^i\right)(Id_F - u) \\ &= \left(Id_F + u + u^2 + \cdots + u^n\right)(Id_F - u) \\ &= Id_F + u + u^2 + \cdots + u^n - \left(Id_F + u + u^2 + \cdots + u^n\right) u \\ &= Id_F + u + u^2 + \cdots + u^n - u - u^2 - \cdots - u^n - u^{n+1} \\ &= Id_F - u^{n+1}\end{aligned}$$

so

$$\begin{aligned}\lim_{n\to\infty} (Id_F - u)\left(\sum_{i=0}^{n} u^i\right) &= (Id_F - u) \lim_{n\to\infty} \sum_{i=0}^{n} u^i \\ &= Id_F\end{aligned}$$

we deduce that $Id_F - u$ is invertible and $(Id_F - u)^{-1} = \sum_{i=0}^{\infty} u^i$. ■

Lemma 2.4.2

The set of bijective linear maps $Isom(E, F)$ is an open subset of $\mathcal{L}(E, F)$.

Proof. Let u be an isomorphism and $h \in \mathcal{L}(E, F)$ with $\|h\|_{\mathcal{L}(E,F)} < \frac{1}{\|u^{-1}\|_{\mathcal{L}(E,F)}}$ then

$$\begin{aligned}u - h &= uId_F - uu^{-1}h \\ &= u\left(Id_F - u^{-1}h\right)\end{aligned}$$

and

$$\|u^{-1}h\|_{\mathcal{L}(E,F)} \leq \|u^{-1}\|_{\mathcal{L}(E,F)}\|h\|_{\mathcal{L}(E,F)} < 1.$$

Applying lemma 2.4.1, we have $Id_F - u^{-1}h$ and $u\,(Id_F - u^{-1}h) = u - h \in Isom(E, F)$ so the open ball $B(u, \frac{1}{\|u^{-1}\|_{\mathcal{L}(E,F)}})$ is contained in $Isom(E, F)$ which means that $Isom(E, F)$ is an open subset of $\mathcal{L}(E, F)$. ■

Theorem 2.4.1

The map associating to $u \in Isom(E, F)$ its inverse $u^{-1} \in Isom(F, E)$ is continuous.

Proof. Take $u \in Isom(E,F)$ and $h \in \mathcal{L}(E,F)$ with $\|h\|_{\mathcal{L}(E,F)} < \frac{1}{\|u\|_{\mathcal{L}(E,F)}}$. We have

$$\begin{aligned} u - h &= uu^{-1}u - hu^{-1}u \\ &= \left(uu^{-1} - hu^{-1}\right)u \\ &= \left(Id_F - hu^{-1}\right)u \end{aligned}$$

so

$$\begin{aligned} (u-h)^{-1} - u^{-1} &= \left(Id_F - hu^{-1}\right)^{-1}u^{-1} - u^{-1} \\ &= \left[\left(Id_F - hu^{-1}\right)^{-1} - Id_F\right]u^{-1}. \end{aligned}$$

By lemma 2.4.1

$$\left(Id_F - hu^{-1}\right)^{-1} = \sum_{n=0}^{\infty}\left(hu^{-1}\right)^n$$

hence

$$\begin{aligned} (u-h)^{-1} - u^{-1} &= \left[\sum_{n=0}^{\infty}\left(hu^{-1}\right)^n - Id_F\right]u^{-1} \\ &= \left[Id_F + \sum_{n=1}^{\infty}\left(hu^{-1}\right)^n - Id_F\right]u^{-1} \\ &= \sum_{n=1}^{\infty}\left(hu^{-1}\right)^n u^{-1}. \end{aligned}$$

Taking norms

$$\begin{aligned} \|(u-h)^{-1} - u^{-1}\| &= \|\sum_{n=1}^{\infty}\left(hu^{-1}\right)^n u^{-1}\| \\ &\leq \|\sum_{n=1}^{\infty}\left(hu^{-1}\right)^n\|\|u^{-1}\| \\ &\leq \sum_{n=1}^{\infty}\|hu^{-1}\|^n\|u^{-1}\| \\ &\leq \|u^{-1}\|\frac{\|hu^{-1}\|}{1-\|hu^{-1}\|} \end{aligned}$$

and then

$$\lim_{\|h\|\to 0}\|(u-h)^{-1} - u^{-1}\| \leq \|u^{-1}\|\lim_{\|h\|\to 0}\frac{\|hu^{-1}\|}{1-\|hu^{-1}\|} = 0$$

hence continuity. ■

§2.4.2 Local Inversion Theorem

Theorem 2.4.2: Local inversion theorem - first version

Let $f : U \to V$ be a homeomorphism. f is a $\mathcal{C}^1-$diffeomorphism if and only if

1. f is of class $\mathcal{C}^1$.
2. $\forall x \in U; d_x f \in Isom(E, F)$ i.e. the differential of f is invertible.

Proof. We start with the first point: if f is a $\mathcal{C}^1-$diffeomorphism then its inverse f^{-1} is also a $\mathcal{C}^1-$diffeomorphism. We have

$$\begin{cases} f \circ f^{-1} & = Id_F \\ f^{-1} \circ f & = Id_E \end{cases}$$

and take $x \in U$ and $y = f(x) \in V$ we know that (see **??**)

$$d_x \left(f^{-1} \circ f\right) = d_{f(x)} \left(f^{-1}\right) \circ d_x f = Id_E$$

and

$$d_{f(x)} \left(f \circ f^{-1}\right) = d_x f \circ d_{f(x)} \left(f^{-1}\right) = Id_F$$

so $d_x f \in Isom(E, F)$.

For the second point, take $a \in U$ and $d_x f \in Isom(E, F)$. By definition

$$\lim_{\|x-a\|_E} \frac{\|f(x) - f(a) - d_x f \bullet (x - a)\|_E}{\|x - a\|_E} = 0.$$

Denoting $b = f(a), y = f(x), (d_x f)^{-1} = \varphi \in Isom(E, F)$ and

$$\psi(x) = \frac{f(x) - f(a) - d_x f \bullet (x - a)}{\|x - a\|_E}.$$

Since φ is linear (inverse of a bijective linear map) we have

$$\begin{aligned} \varphi\left(\psi(x)\|x - a\|_E\right) &= \varphi\left(f(x) - f(a) - d_x f \bullet (x - a)\right) \\ &= \varphi \circ f(x) - \varphi \circ f(a) - [\varphi \circ d_x f] \bullet (x - a) \\ &= \varphi \circ f(x) - \varphi \circ f(a) - (x - a) \\ &= \varphi(y) - \varphi(b) - \left(f^{-1}(y) - f^{-1}(b)\right) \\ &= -\left[f^{-1}(y) - f^{-1}(b) - \varphi(b - a)\right] \end{aligned}$$

taking norms

$$\frac{\|f^{-1}(y) - f^{-1}(b) - \varphi(b - a)\|_E}{\|y - b\|_F} = \frac{\|x - a\|_E}{\|f(x) - f(a)\|_F} \|\varphi \circ \psi(x)\|_E$$

and since f is differentiable and knowing that $\lim_{x\to a}\psi(x)=\lim_{\|x-a\|_E\to 0}\psi(x)=0$ we obtain

$$\begin{aligned}\lim_{y\to b}\frac{\|f^{-1}(y)-f^{-1}(b)-\varphi(b-a)\|_E}{\|b-a\|_E} &= \lim_{x\to a}\frac{\|x-a\|_E}{\|f(x)-f(a)\|_F}\|\varphi\circ\psi(x)\|_E \\ &= 0\end{aligned}$$

so f^{-1} is differentiable at $b=f(a)$ and its differential is $d_{f(a)}\left(f^{-1}\right)=\varphi=(d_x f)^{-1}$. ■

Theorem 2.4.3: Fixed Point Theorem

Let E be a Banach space and $f:E\to E$ a $K-$Lipschitz map with $0<K<1$ (in this case we say f is a contracting function), then there exists a unique $x\in E$ such that $f(x)=x$ called the fixed point of f.

Proof. The proof is in two steps: existence and uniqueness.

1. Let $x_0\in E$. Define the sequence

$$x_{n+1}=f(x_n)=f^{n+1}(x_0);n\geq 1.$$

We have

$$\begin{aligned}\|x_{n+1}-x_n\|_E &= \|f(x_n)-f(x_{n-1})\|_E \\ &\leq K\|x_n-x_{n-1}\|_E \\ &\vdots \\ &\leq K^n\|x_1-x_0\|_E.\end{aligned}$$

Then

$$\begin{aligned}\|x_{n+p}-x_n\|_E &= \|\sum_{i=1}^{p}\left(x_{n+i}-x_{n+i-1}\right)\|_E \\ &\leq \sum_{i=1}^{p}\|x_{n+i}-x_{n+i-1}\|_E \\ &\leq \sum_{i=1}^{p}K^{n+i-1}\|x_1-x_0\|_E \\ &= K^n\frac{1-K^p}{1-K}\|x_1-x_0\|_E \\ &\leq \frac{K^n}{1-K}\|x_1-x_0\|_E\end{aligned}$$

which means that $\{x_n\}_n$ is a Cauchy sequence in the complete Banach space E, so $\exists x \in E$ such that

$$\lim_{n\to\infty} x_n = x$$

and by continuity of f

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \\ &= f(x) \\ &= x.\end{aligned}$$

This proves the existence of the fixed point x.

2. Let $x, y \in E$ such that $x = f(x)$ and $y = f(y)$. We have

$$\begin{aligned}\|x - y\|_E &= \|f(x) - f(y)\|_E \\ &\leq K\|x - y\|_E\end{aligned}$$

but since $0 < K < 1$ we necessarily have $\|x - y\|_E = 0$ and $x = y$.

■

Theorem 2.4.4: Local Inversion Theorem - second version

Let E, F be two Banach spaces, U an open subset of E and $f : U \to F$ of class $\mathcal{C}^1$. If we have $a \in U$ such that $d_a f \in Isom(E, F)$, then there exists an open neighborhood of a denoted V and an open set $W \subset F$ such that $f : V \to W$ is a diffeomorphism of class $\mathcal{C}^1$.

To prove this result we will need the following lemmas.

Lemma 2.4.3

Under the conditions of theorem 2.4.4 and if we define the function ψ by

$$\begin{aligned}\psi : U &\to E \\ x &\mapsto x - (d_a f)^{-1} \circ f(x)\end{aligned}$$

then

1. ψ is of class $\mathcal{C}^1$ on U.

2. $\forall K \in]0,1[\,, \exists r > 0$ such that ψ is $K-$Lipschitz on the open ball $B(a,r)$.

Proof. Computing the differential of function ψ at point a

$$\begin{aligned} d_a\psi &= Id_E - (d_af)^{-1} \circ d_af \\ &= Id_E - Id_E \\ &= 0. \end{aligned}$$

Since ψ is of class $\mathcal{C}^1$ then its differential is continuous so for any $K \in]0,1[$ there exists $r > 0$ such that

$$\|d_x\psi\| \leq K, \forall x \in B(a,r) \subset U.$$

Applying the mean value theorem 2.3.3 we have on $B(a,r)$

$$\|\psi(y) - \psi(x)\|_E \leq K\|y - x\|_E.$$

■

Lemma 2.4.4

The map $\Phi = x - \psi$:

$$\begin{aligned} \Phi : U &\to E \\ x &\mapsto (d_af)^{-1} \circ f(x) \end{aligned}$$

is differentiable and injective on $B(a,r)$.

Proof. Let $x, y \in B(a,r)$ then

$$\begin{aligned} \|\Phi(y) - \Phi(x)\|_E &= \|\,(\Phi(y) - y) - (\Phi(x) - x) + (y - x)\,\|_E \\ &\geq \|y - x\|_E - \|\,(\Phi(y) - y) - (\Phi(x) - x)\,\|_E \\ &\geq \|y - x\|_E - \|\psi(x) - \psi(y)\|_E \\ &\geq \|y - x\|_E - K\|y - x\|_E \\ &= (1 - K)\,\|y - x\|_E \end{aligned}$$

and so if $\|\Phi(y) - \Phi(x)\|_E = 0 \implies \|y - x\|_E = 0$ so Φ is injective. The differentiability of Φ follows from that of f. ■

Lemma 2.4.5

Let $b = \Phi(a)$. If $y \in B(b, (1-K)r)$ then there exists a unique $x \in B(a,r)$ such that $y = \Phi(x)$.

Proof. The existence of x follows from the existence of a solution to the equation
$$y = \Phi(x) = x - \psi(x).$$
Define
$$\begin{aligned} g_y : \overline{B}(a,r) &\to \overline{B}(a,r) \\ x &\mapsto y + \psi(x) \end{aligned}$$
then we have
$$\begin{aligned} \|g_y(x) - a\|_E &= \|y + \psi(x) - a\|_E \\ &= \|y + \psi(x) - a + b - b\|_E \\ &\leq \|y - b\|_E + \|\psi(x) - a + b\|_E \\ &= \|y - b\|_E + \|\psi(x) - (a - \Phi(a))\|_E \\ &= \|y - b\|_E + \|\psi(x) - \psi(a)\|_E \\ &\leq \|y - b\|_E + K\|x - a\|_E \\ &\leq (1-K)r + Kr \\ &= r \\ &\leq (1-K)\|y - x\|_E \end{aligned}$$
so for all $x \in \overline{B}(a,r)$ the image $g_y(x) \in \overline{B}(a,r)$. On the other hand, if $x_1, x_2 \in \overline{B}(a,r)$ we have
$$\begin{aligned} \|g_y(x_2) - g_y(x_1)\|_E &= \|\psi(x_2) - \psi(x_1)\|_E \\ &\leq K\|x_2 - x_1\|_E \end{aligned}$$
so the function g_y is contracting on the closed set $\overline{B}(a,r)$ of a Banach space so by the fixed point theorem 2.4.3, $\exists ! x \in \overline{B}(a,r)$ such that
$$g_y(x) = y + \psi(x) = xeq : 2.7 \tag{2.4}$$
hence
$$y = \Phi(x).$$
■

Lemma 2.4.6

The set $V = \Phi^{-1}(B(b,(1-K)r))$ is open in $B(a,r)$ and the restriction of Φ from V to $B(b,(1-K)r)$ is a homeomorphism.

Proof. Define the function

$$\begin{aligned} g : B(b,(1-K)r) &\to B(a,r) \\ y &\mapsto g(y) = x = y + \psi(x) \end{aligned}$$

which associates to y the fixed point of function g_y. From formula **??** we deduce that

$$x = g(y) \Longleftrightarrow y = \Phi(x)$$

and so $g = \Phi^{-1}$ hence

$$V = \Phi^{-1}\left(B(b,(1-K)r)\right) = g\left(B(b,(1-K)r)\right)$$

is an open subset of $B(a,r)$ and the uniqueness of x ensures the injectivity of g and so the map

$$g : B(b,(1-K)r) \to V$$

is bijective and its inverse is Φ. Now, we check the continuity of Φ: let $x_1, x_2 \in V$ and $y_1, y_2 \in B(b,(1-K)r)$ such that $x_1 = g(y_1)$ and $x_2 = g(y_2)$ then

$$\begin{aligned} \|g(y_1) - g(y_2)\|_E &= \|x_1 - x_2\|_E \\ &= \|x_1 - x_2\|_E \\ &= \|\,[y_1 + \psi(x_1)] - [y_2 + \psi(x_2)]\,\|_E \\ &\leq \|\psi(x_1) - \psi(x_2)\|_E + \|y_1 - y_2\|_E \\ &\leq K\,\|x_1 - x_2\|_E + \|y_1 - y_2\|_E \end{aligned}$$

so

$$\begin{aligned} \|g(y_1) - g(y_2)\|_E - K\,\|g(y_1) - g(y_2)\|_E &\leq \|y_1 - y_2\|_E \\ \|g(y_1) - g(y_2)\|_E - K\,\|g(y_1) - g(y_2)\|_E &\leq \|y_1 - y_2\|_E \\ (1-K)\|g(y_1) - g(y_2)\|_E &\leq \|y_1 - y_2\|_E \\ \|g(y_1) - g(y_2)\|_E &\leq \frac{1}{1-K}\,\|y_1 - y_2\|_E \end{aligned}$$

so g is a $\frac{1}{1-K}-$Lipschitz function and *a fortiori* continuous and we have

$$\Phi : V \to B(b,(1-K)r)$$

is a homeomorphism. ■

Now, let's proceed to the proof of theorem 2.4.4.

Proof. By lemma 2.4.6, the map

$$\Phi = g^{-1} : V \to B(b, (1-K)r)$$

is a homeomorphism of class $\mathcal{C}^1$ and

$$\begin{aligned} d\Phi : V &\to B(b, (1-K)r) \\ x &\mapsto d_x\Phi = Id_E - d_a\psi = Id_E \end{aligned}$$

so the differential of Φ is a continuous map and $d\Phi \in Isom(E, F)$. And as we showed that $Isom(E, F)$ is an open subset of $\mathcal{L}(E, F)$ (lemma 2.4.2) then there exists an open neighborhood $\tilde{V}$ of a such that $\forall x \in \tilde{V}; d_x\Phi \in Isom(E, F)$. By theorem 2.4.4, we deduce that

$$\Phi : \tilde{V} \to \Phi(\tilde{V})$$

is a diffeomorphism of class $\mathcal{C}^1$ and denoting $f = d_af \circ \Phi : \tilde{V} \to W = [d_af \circ \Phi](\tilde{V})$ which is also a diffeomorphism of class $\mathcal{C}^1$ in the open set W of F. ■

This theorem means that a function is locally invertible in a neighborhood of a point where its differential is invertible. If E and F are finite-dimensional Banach spaces then we have:

Corollary 2.4.1

f $f : E \to F$ is a map of class $\mathcal{C}^1$, then f is a local diffeomorphism at $x \in E$ *iff*

$$\det(J_x f) \neq 0.$$

§2.4.3 Implicit Function Theorem

Let E, F and G be Banach spaces and

$$\begin{aligned} f : E \times F &\to G \\ (x, y) &\mapsto z = f(x, y) \end{aligned}$$

and consider the equation (also called level curve) $g(x, y) = 0_G$. Our goal is to write y as a function of x.

Theorem 2.4.5

Let E, F and G be Banach spaces and U an open subset of $E \times F$. Let $(x_0, y_0) \in U$ and $f : U \to G$ a map of class $\mathcal{C}^1$ such that $f(x_0, y_0) = 0_G$. If the partial differential $\frac{\partial f}{\partial y}(x_0, y_0) \in Isom(F, G)$ i.e. it is invertible then

1. There exists an open neighborhood V of (x_0, y_0) in U,
2. There exists an open neighborhood W of x_0 in E,
3. and a function $\varphi : W \to F$ of class $\mathcal{C}^1$,

$$(x, y) \in V, f(x, y) = 0_G \Leftrightarrow x \in W, y = \varphi(x).$$

Remark 2.3. *$\frac{\partial f}{\partial y}(x_0, y_0)$ is the differential of the map $y \mapsto f(x_0, y)$ at point y_0.*

For the case of vector-valued functions of vector variables, we have the analogous result

Corollary 2.4.2

et f be a function of class $\mathcal{C}^k$ defined on an open set U of $\mathbb{R}^p \times \mathbb{R}^q$, with values in $\mathbb{R}^q$. Let $(x_0, y_0) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $f(x_0, y_0) = 0$. Assume that the matrix

$$\left(\frac{\partial f_i}{\partial y_j}(x_0, y_0)\right)_{1 \leq i \leq q, 1 \leq j \leq q}$$

is invertible. Then there exists an open neighborhood V of (x_0, y_0) in $\mathbb{R}^p \times \mathbb{R}^q$, a neighborhood W of x_0 in $\mathbb{R}^p$ and a function $g : W \to \mathbb{R}^q$ of class $\mathcal{C}^k$ such that, for all $(x, y) \in V$, we have

$$f(x, y) = 0 \Longleftrightarrow y = g(x).$$

Proof of theorem 2.4.5. We will apply the local inversion theorem to the function

$$\begin{aligned} g : U &\to E \times G \\ (x, y) &\mapsto \psi(x, y) = (x, f(x, y)). \end{aligned}$$

Clearly, the function g is of class $\mathcal{C}^1$ and we have $g(x_0 y_0) = (x_0, 0_G)$, as well as the invertible differential (linear) map

$$d_{(x_0,y_0)} g \bullet (h, k) = \begin{pmatrix} h \\ \frac{\partial f}{\partial x}(x_0, y_0) \bullet h + \frac{\partial f}{\partial y}(x_0, y_0) \bullet k \end{pmatrix}^T$$

and

$$d_{(x_0,y_0)} g = \begin{pmatrix} Id_E & 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \in Isom(E \times F, E \times G)$$

with inverse

$$\left(d_{(x_0,y_0)}g\right)^{-1} = \begin{pmatrix} Id_E & 0 \\ -\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} \frac{\partial f}{\partial x}(x_0,y_0) & \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} \end{pmatrix}$$

so $\forall (h',k') \in E \times G$:

$$d_{(x_0,y_0)}g \bullet (h,k) = (h',k') = (h, \frac{\partial f}{\partial x}(x_0,y_0) \bullet h + \frac{\partial f}{\partial y}(x_0,y_0) \bullet k)$$

which is equivalent to saying that

$$\begin{cases} h & = h' \\ k & = \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} \bullet \left(k' - \frac{\partial f}{\partial x}(x_0,y_0) \bullet h'\right) \end{cases}$$

which is well-defined so we deduce that $d_{(x_0,y_0)}g$ is invertible and we have

$$\begin{aligned} \left(d_{(x_0,y_0)}g\right)^{-1} &: E \times G \to E \times F \\ (h',k') &\mapsto (h', \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} \bullet \left(k' - \frac{\partial f}{\partial x}(x_0,y_0) \bullet h'\right) \end{aligned}$$

By theorem 2.4.4, there exists an open set $V \subset E \times F$ neighborhood of (x_0, y_0), and a neighborhood of $(x_0, 0)$ which we can assume to be of the form $W \times Z_{0_G}$ so W is an open neighborhood of x_0 and Z_{0_G} a neighborhood of 0_G such that the map $g : V \to W \times Z_{0_G}$ is a diffeomorphism of class $\mathcal{C}^1$. Now construct the inverse map of g:

$$\begin{aligned} g^{-1} : W \times Z_{0_G} &\to V \\ (x,z) &\mapsto g^{-1}(x,z) \\ &= (x, \phi(x,z)) \end{aligned}$$

where the map ϕ is of class $\mathcal{C}^1$. In other words,

$$((x,y) \in V \text{ and } f(x,y) = 0) \iff ((x,z) \in W \times Z_{0_G} \text{ and } y = \phi(x,z))$$

and in particular

$$((x,y) \in V \text{ and } f(x,y) = 0) \iff (x \in W \text{ and } y = \phi(x,0) = \varphi(x)) \, .$$

■

Exercise 2.3. *Let the following function*

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ (x,y,z) &\mapsto (x^2+y^2+z^2)\ln(x+y+z) - e^{x+y} - 1 \end{aligned}$$

1. *Show that the partial derivatives exist at every point of its domain and write explicitly* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *and* $\frac{\partial f}{\partial z}$.

2. *Show that on the surface given by* $f(x,y,z)=0$ *there exists a neighborhood* V *of the origin* $(0,0)$, *a neighborhood* W *of* 1 *and a function* $\phi : V \to W$ *of class* $\mathcal{C}^\infty$ *such that* $\phi(0,0)=1$ *and* $f(x,y,\phi(x,y)) = 0, \forall (x,y) \in V$.

3. *Compute the partial derivatives of* ϕ *at point* $(0,0)$.

Solution. *We know that the functions* $(x,y,z) \mapsto x^2+y^2+z^2$, $(x,y,z) \mapsto \ln(x+y+z)$ *and* $(x,y) \mapsto e^{x+y}$ *are of class* $\mathcal{C}^\infty$ *on their domains of definition so* f *is also, which ensures the existence of the partial derivatives*

$$\frac{\partial f}{\partial x}(x,y,z) = 2x\ln(x+y+z) + \frac{x^2+y^2+z^2}{x+y+z} - e^{x+y}$$
$$\frac{\partial f}{\partial y}(x,y,z) = 2y\ln(x+y+z) + \frac{x^2+y^2+z^2}{x+y+z} - e^{x+y}$$
$$\frac{\partial f}{\partial z}(x,y,z) = 2z\ln(x+y+z) + \frac{x^2+y^2+z^2}{x+y+z}$$

We have $f(0,0,1)=0$ *and* $\frac{\partial f}{\partial z}(0,0,1) = 1 \neq 0$ *so* $d_{(0,0,1}f$ *is invertible and by the implicit function theorem there exists a neighborhood* V *of* $(0,0)$, *a neighborhood* W *of* 1 *and a function* $\phi : V \to W$ *of class* $\mathcal{C}^\infty$ *such that*

$$\begin{cases} \phi(0,0) & = 1 \\ f(x,y,\phi(x,y)) & = 0. \end{cases}$$

To compute the first partial derivatives of ϕ *at* $(0,0)$, *we differentiate the equation* $f(x,y,\phi(x,y))=0$ *with respect to* x *and* y:

$$\frac{\partial f}{\partial x}(x,y,\phi(x,y))\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}(x,y,\phi(x,y))\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}(x,y,z)\frac{\partial \phi}{\partial x}(x,y) = 0$$
$$\implies \frac{\partial f}{\partial x}(x,y,\phi(x,y)) + \frac{\partial \phi}{\partial x}(x,y)\frac{\partial f}{\partial z}(x,y,z) = 0$$

and

$$\frac{\partial f}{\partial y}(x,y,\phi(x,y))\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}(x,y,\phi(x,y))\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}(x,y,z)\frac{\partial \phi}{\partial y}(x,y) = 0$$
$$\implies \frac{\partial f}{\partial y}(x,y,\phi(x,y)) + \frac{\partial f}{\partial z}(x,y,z)\frac{\partial \phi}{\partial y}(x,y) = 0.$$

For $(x,y,z) = (0,0,1)$ *we obtain*

$$\begin{cases} \frac{\partial f}{\partial x}(0,0,1) + \frac{\partial \phi}{\partial x}(0,0)\frac{\partial f}{\partial z}(0,0,1) & = 0 \\ \frac{\partial f}{\partial y}(0,0,1) + \frac{\partial f}{\partial z}(0,0)\frac{\partial \phi}{\partial y}(0,0,1) & = 0. \end{cases}$$

hence

$$\begin{cases} \frac{\partial \phi}{\partial x}(0,0) & = 0 \\ \frac{\partial \phi}{\partial y}(0,0) & = 0. \end{cases}$$

§2.5 Exercices

Exercice 2.1. *Show that if the differential of a function $f : E \longrightarrow F$ at a point a, if it exists, is unique.p:oàkàungf s*

Exercice 2.2. *Study the existence of the derivative of the function $f : (x, y) \mapsto xy^2$ in the direction of the vector $v = (1, -2)$ at the point $A = (2, 1)$. Determine its value if it exists.*

Exercice 2.3. *Justify that the following applications are differentiable and explicitly determine their differentials.*

1. $f : \mathcal{M}n(\mathbb{R}) \longrightarrow \mathcal{M}n(\mathbb{R})$ *and* $M \longmapsto M^2$.
2. $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.
3. $f \mapsto f^{-1}$ *from* $GL(E)$ *to itself, where* E *is a finite-dimensional real vector space.*

Exercice 2.4. *Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $A : (x, y, z) \mapsto (x, \sin, y, y, \sin, x, z)$. Justify the existence and calculate $\operatorname{div}(A)$, $\operatorname{rot}(A)$, and $\nabla(\operatorname{div}(A))$.*

Reminder: $div(A) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$, $rot(A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$.

Exercice 2.5. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function.*

1. *We define*

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto f(2 + 2t, t^2). \end{aligned}$$

Show that $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and calculate its derivative in terms of the partial derivatives of f.

2. *The same questions for:*

$$\begin{aligned} h : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (u, v) &\longmapsto f(uv, u^2 + v^2) \end{aligned}$$

Exercice 2.6. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by:*

$$f : (x, y) \mapsto \begin{cases} \frac{x^2y-y^3}{x^2+y^2}, & si\ (x, y) \neq (0, 0), \\ 0, & at\ (0, 0). \end{cases}$$

1. *Calculate the partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *at* $(0, 0)$.

2. *Is the function* f *differentiable at* $(0, 0)$*?*

Exercice 2.7. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by:*

$$f : (x, y) \mapsto \begin{cases} \frac{xy^2}{x^2+y^2}, & si\ (x, y) \neq (0, 0), \\ 0, & at\ (0, 0). \end{cases}$$

Show that f *has directional derivatives in all directions at* $(0, 0)$ *but is not differentiable there.*

Exercice 2.8. *Let* E *and* F *be two vector spaces over* $\mathbb{R}$ *of finite dimensions, and let* $f : E \to F$ *be a* C^1 *function. In the following cases, find the dimension of the Jacobian matrix of* f.

1. f *is a real function of a real variable* ($E = F = \mathbb{R}$).

2. f *is a vector-valued function of a real variable* ($E = \mathbb{R},\ F = \mathbb{R}^p$).

3. f *is a real function of a vector variable* ($E = \mathbb{R}^n,\ F = \mathbb{R}$).

4. f *is a vector-valued function of a vector variable* ($E = \mathbb{R}^n,\ F = \mathbb{R}^p$).

Using the coefficients of the Jacobian matrix of f, *express the differential of* f *at any point in* E.

Exercice 2.9. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable function. What is the relationship between the differential of* f *at* $a \in \mathbb{R}^n$ *and the gradient of* f *at the same point?*

Exercice 2.10. *Let* $f : (\mathbb{R}_+)^2 \to \mathbb{R}$, *defined by* $f : (x, y) \mapsto 2x + 5y + x^2(\sqrt{y} + \sqrt{x})$.

1. *At which points is* f *continuous?*

2. *At which points does* f *have partial derivatives?*

3. *On what set is* f C^1*?*

4. *At which points is* f *differentiable?*

5. *At which points does* f *have directional derivatives?*

Exercice 2.11. *Consider the function:*

$$f : (x, y) \mapsto \begin{cases} xy \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & si\ (x, y) \neq (0, 0), \\ 0, & at\ (0, 0). \end{cases}$$

1. *Show that f has partial derivatives at every point in $\mathbb{R}^2$ and calculate them.*

2. *Show that f is not $\mathcal{C}^1$ on $\mathbb{R}^2$.*

3. *Show that f is differentiable at the point $(0, 0)$.*

Exercice 2.12. *Justify the differentiability of the following functions and calculate their Jacobian matrices.*

1. $f : (x, y) \mapsto e^{xy}(x + y)$,
2. $g : (x, y, z) \mapsto xy + yz + zx$,
3. $h : (x, y) \mapsto (y \sin x, \cos x)$.

Exercice 2.13. *Justify the differentiability of the following functions and calculate their differentials.*

1. $f : (x, y, z) \mapsto \left(\frac{x^2-z^2}{2}, \sin x \sin y\right)$,
2. $g : (x, y) \mapsto \left(xy, \frac{x^2}{2} + y, \ln(1 + x^2)\right)$.

Exercice 2.14. *Let f be a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}^2$. We define $g : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ as follows :*

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \mathbb{R},\ g(r, \theta) = f(r \cos(\theta), r \sin(\theta)).$$

We need to show that g is differentiable and express $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$ in terms of the partial derivatives of f.

Exercice 2.15. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function.*

1. *Determine the derivative of $u : x \mapsto f(x, -x)$ and the differential of $g : (x, y) \mapsto f(y, x)$.*

2. *Let E and F be two normed vector spaces, U be an open set in E, and $f : U \to F$ be a differentiable function. For any $a \in U$ and $v \in E$, find the derivative of the function $t \mapsto f(a + tv)$ at $t = 0$.*

Exercice 2.16. *Let x and y be two differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and let f be a $\mathcal{C}^1$ function from $\mathbb{R}^2$ to $\mathbb{R}$. Consider $z : t \mapsto f(x(t), y(t))$. We want to determine z', the derivative of z with respect to t.*

Apply this formula to the following specific cases :

1. *$f(x, y) = x^2 + 2xy + 4y^2$, $x(t) = t$, and $y(t) = e^{3t}$.*

2. $f(x,y) = xy^2 + x^2y$, $x(t) = t^2$, *and* $y(t) = \ln t$.

Exercice 2.17. *Consider the functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *and* $g : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by:*

$$\forall\ (x,y) \in \mathbb{R}^2,\ f(x,y) = x^2 - y^2 \quad \text{and} \quad \forall\ (x,y,z) \in \mathbb{R}^3,\ g(x,y,z) = (x+y+z, x-y+z)$$

1. *Let* $h = f \circ g$. *Determine* h. *Show that* f, g, *and* h *are* C^1 *functions and write their Jacobians.*

2. *Verify that* $J_h(x,y,z) = J_f(g(x,y,z))J_g(x,y,z)$, *where* J_f, J_g, *and* J_h *denote the Jacobian matrices of* f, g, *and* h *respectively.*

Exercice 2.18. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a differentiable function. We define* $f : \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}$ *by:*

$$\forall (x,y) \in \mathbb{R}^* \times \mathbb{R},\ f(x,y) = \varphi\left(\frac{y}{x}\right).$$

Justify that the quantity $x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y)$ *makes sense for all* $(x,y) \in \mathbb{R}^* \times \mathbb{R}$, *and calculate it.*

Exercice 2.19. *Let* U *be an open subset of* $\mathbb{R}^n$ *and* $f : U \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *function that does not vanish. Show that the inverse function* $\frac{1}{f}$ *is also* $\mathcal{C}^1$ *and give its differential at every point in* U.

Chapter 3

Constant Rank Theorem

§3.1 Rank of a map

Definition 3.1.1

Let $f : U \subset E \to F$ be a function of class $\mathcal{C}^1$. Assume that the image of the differential at point $x \in U$

$$Im\left(d_x f\right) := \{d_x f \bullet h, h \in E\}$$

is a vector subspace of E. We then say that the function f has finite rank at x and we denote

$$\text{rank}_x f = \dim\left(Im\left(d_x f\right)\right).$$

Remark 3.1. *If the domain space is finite-dimensional with* $\dim(E) = n < \infty$ *then*

$$rank_x f = n - \dim\left(Ker\left(d_x f\right)\right).$$

Remark 3.2. *If* $\dim(E) = n$ *and* $\dim(F) = m$ *and* $f \in \mathcal{C}^1(E, F)$ *with* $rank_x f = p \in \mathbb{N}$ *then there exists* $\{e_1, e_2, \cdots, e_n\}$ *a basis of* E *and* $\{\overline{e_1}, \overline{e_2}, \cdots, \overline{e_m}\}$ *a basis of* F *such that*

$$\{d_x f \bullet e_1, d_x f \bullet e_2, \cdots, d_x f \bullet e_p\}$$

is a basis of $Im\left(d_x f\right)$ *and*

$$\{e_{p+1}, e_{p+2}, \cdots, e_m\}$$

is a basis of $Ker\left(d_x f\right)$. *The matrix associated to* $d_x f$ *denoted* $M_x f$ *will be written*

$$M_x f = \begin{pmatrix} Id_{p\times p} & 0 \\ 0 & 0 \end{pmatrix}$$

and so we can say that the rank of a function f *is the rank of the largest submatrix extracted from the matrix associated to its differential* $M_x f$ *that is invertible.*

Definition 3.1.2

We say that the function $f \in \mathcal{C}^1(U \subset E, F)$ is an *immersion* at $x \in U$ if the differential $d_x f$ is injective$\Longleftrightarrow Ker(d_x f) = \{0\}$.

Definition 3.1.3

We say that the function $f \in \mathcal{C}^1(U \subset E, F)$ is a *submersion* at $x \in U$ if the differential $d_x f$ is surjective$\Longleftrightarrow Im(d_x f) = F$.

Remark 3.3. *From remark 3.1 we have*

$$\begin{cases} f \textit{ is an immersion} & \Leftrightarrow rank_x f = n \\ f \textit{ is a submersion} & \Leftrightarrow rank_x f = m \end{cases}.$$

If f *has* $rank_x f = p$ *then there exists* $\{i_1, i_2, \cdots, i_p\} \subseteq \{1, 2, \cdots, n\}$ *and* $\{j_1, j_2, \cdots, j_p\} \subseteq \{1, 2, \cdots, m\}$ *such that*

§3.2 Submersion characterization theorem

Theorem 3.2.1

Let $f : U \in \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a submersion of class $\mathcal{C}^1$ at $x_0 \in U$ then there exists

1. An open neighborhood $V \subset U$ of x_0.
2. An open set $W \subset \mathbb{R}^n$.
3. A $\mathcal{C}^1-$diffeomorphism $g : V \longrightarrow W$

such that

$$f \circ g^{-1} : W \longrightarrow \mathbb{R}^m$$
$$y = (y_1, \cdots, y_m, \cdots, y_n) \longmapsto (y_1, \cdots, y_m)$$

is a canonical projection.

Proof. We know that

$$\begin{aligned} f : U &\longrightarrow \mathbb{R}^m \\ (x_1, \cdots, x_n) &\longmapsto \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{pmatrix} \end{aligned}$$

is a submersion at point $x_0 \Leftrightarrow \mathrm{rank}_{x_0} f = m$. We can assume that

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{vmatrix} \neq 0.$$

We set (since $n \geq m$)

$$\begin{aligned} h : U &\longrightarrow \mathbb{R}^n \\ x = (x_1, \cdots, x_m, \cdots, x_n) &\longmapsto \begin{pmatrix} f(x) \begin{cases} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{cases} \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

In this case

$$d_{x_0} h = \begin{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} & \begin{pmatrix} \frac{\partial f_1}{\partial x_{m+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_{m+1}} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \end{pmatrix}$$

so

$$\det\left(d_{x_0 h}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{vmatrix}$$

and by the local inversion theorem, there exists an open neighborhood $V \subset U$ of x_0 and an open neighborhood $W \subset \mathbb{R}^n$ of $h(x_0)$ such that $h : V \longrightarrow W$ is a $\mathcal{C}^1-$diffeomorphism. ■

- For $x \in V \subset \mathbb{R}^n$ and $y \in W \subset \mathbb{R}^m$ with $y = h(x)$ we have

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix}$$

  so by correspondence $y_i = f_i(x), \forall i = 1, 2, \cdots, m$.

- If $g = h : V \longrightarrow W$ then

$$\begin{aligned}(y_1, \cdots, y_m) &= f(x) \\ &= f(h^{-1}(y) \\ &= f(g^{-1}(y) \\ &= f \circ g^{-1}(y)\end{aligned}$$

  is a canonical projection.

Exercise 3.1. *Let*

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto xe^y \end{aligned}$$

1. Define a $\mathcal{C}^1-$diffeomorphism $g : V \subset \mathbb{R}^2 \longrightarrow W \subset \mathbb{R}^2$ such that $f \circ g^{-1}$ is a canonical projection.

Solution. *We know that the differential*

$$d_{(x,y)}f = e^y \begin{pmatrix} 1 \\ x \end{pmatrix}$$

has rank 1. We set

$$\begin{aligned} g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (f(x, y), y) \end{aligned}$$

and so

$$d_{(x,y)}g = \begin{pmatrix} e^y & xe^y \\ 0 & 1 \end{pmatrix}$$

and $\forall y \in \mathbb{R}$

$$\det\left(d_{(x,y)}g\right) = e^y \neq 0.$$

Let's check if g is injective: let (x, y) and (x', y') be two elements of $\mathbb{R}^2$.

$$\begin{aligned} g(x,y) &= (f(x,y), y) \\ g(x',y') &= (f(x',y'), y') \end{aligned}$$

and if $g(x,y) = g(x',y')$ then

$$\begin{aligned} f(x,y) &= f(x',y') \\ y &= y' \end{aligned}$$

in other words

$$xe^y = x'e^y \Rightarrow x = x'$$

so g is injective and therefore it is a $\mathcal{C}^1-$diffeomorphism. We can write

$$\begin{aligned} g^{-1} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (xe^{-y}, y) \end{aligned}$$

and

$$\begin{aligned} f \circ g^{-1}(x,y) &= f\left((xe^{-y}, y)\right) \\ &= xe^{-y}e^y \\ &= x \end{aligned}$$

which is the canonical projection onto $\mathbb{R}$.

Exercise 3.2. *The same question for the function*

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto (x, z\sin(y)). \end{aligned}$$

Solution. *We have*

$$d_{(x,y,z)}f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z\cos(y) & \sin(y) \end{pmatrix}.$$

We have three cases:

- $z = 0$ and $y = k\pi, k \in \mathbb{Z}$. In this case

$$\begin{aligned} z\cos(y) &= 0 \\ \sin(y) &= 0 \end{aligned}$$

  and $\text{rank}_{(x,y,z)} f = 1 \neq \dim\left(\mathbb{R}^2\right)$ so f is not a submersion.

- If $y \neq k\pi, k \in \mathbb{Z}$ then $\sin(y) \neq 0$ and

$$\begin{vmatrix} 1 & 0 \\ 0 & \sin(y) \end{vmatrix} \neq 0.$$

Set $g(x, y, z) = (x, y, z\sin(y))$ and

$$d_{(x,y,z)}g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z\cos(y) & \sin(y) \end{pmatrix}$$

and

$$\det\left(d_{(x,y,z)}g\right) = -\sin(y) \neq 0$$

so g is a $\mathcal{C}^1-$diffeomorphism and if

$$(x, y, z\sin(y)) = (\alpha, \beta, \gamma) \Rightarrow \begin{cases} \alpha & = x \\ \beta & = y \\ \gamma & = z\sin(\beta) \end{cases}$$
$$\Rightarrow z = \frac{\gamma}{\sin(\beta)}$$

so

$$g^{-1}(x, y, z) = (x, y, \frac{z}{\sin(y)})$$
$$f \circ g^{-1}(x, y, z) = (x, z).$$

- If $z \neq 0$ and $y \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ then

$$\det\begin{pmatrix} 1 & 0 \\ 0 & z\cos(y) \end{pmatrix} = z\cos(y) \neq 0.$$

Set

$$g(x, y, z) = (x, z\sin(y), z)$$

and

$$d_{(x,y,z)}g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z\cos(y) & \sin(y) \\ 0 & 0 & 1 \end{pmatrix}$$
$$\Rightarrow \det\left(d_{(x,y,z)}g\right) = z\cos(y) \neq 0$$

and g is a $\mathcal{C}^1-$diffeomorphism. In a neighborhood of (x, y, z) we have

$$(x, z\sin(y), z) = (\alpha, \beta, \gamma) \Rightarrow \begin{cases} \alpha &= x \\ \beta &= \gamma \sin(y) \\ \gamma &= z \end{cases}.$$

We write

$$\begin{aligned} g^{-1}(x, y, z) &= \left(x, \sin^{-1}\left(\frac{y}{z}\right), z\right) \\ f \circ g^{-1}(x, y, z) &= f\left[x, \sin^{-1}\left(\frac{y}{z}\right), z\right] \\ &= \left(x, z\sin\left(\sin^{-1}\left(\frac{y}{z}\right)\right)\right) \\ &= (x, y). \end{aligned}$$

§3.3 Immersion characterization theorem

Theorem 3.3.1

Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (with $m \geq n$) be a function of class $\mathcal{C}^1$ and an immersion at $x_0 \in U$ then there exists

1. a neighborhood $V \subset U$ of x_0,
2. a neighborhood $W' \subset \mathbb{R}^m$ of $f(x_0)$,
3. an open set $W' \subset \mathbb{R}^m$,
4. and a $\mathcal{C}^1-$diffeomorphism $g : W \longrightarrow W'$

such that

$$\begin{aligned} g \circ f : V \subset \mathbb{R}^n &\longrightarrow W \subset \mathbb{R}^m \\ x = (x_1, \cdots, x_n) &\longmapsto \left(\underbrace{\underbrace{x_1, \cdots, x_n}_{n}, 0, \cdots, 0}_{m}\right) \end{aligned}$$

is a canonical injection.

Proof. Since f is an immersion we have $\text{rank}_{x_0} f = n$. We can assume that

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \neq 0.$$

Set

$$\begin{aligned} h : U \times \mathbb{R}^{m-n} &\longrightarrow \mathbb{R}^m \\ \overline{x} = (x_1, \cdots, x_n, \cdots, x_m) &\longmapsto (f(x), f_{n+1}(x) + x_{n+1}, \cdots, f_m(x) + x_m). \end{aligned}$$

In this case $\overline{x_0} = \left(\underbrace{\underbrace{x_0}_{n}, 0, \cdots, 0}_{m} \right)$, $h(\overline{x_0}) = h(x_0, 0)$ and

$$d_{(x_0,0)} h = \begin{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{\partial f_{n+1}}{\partial x_1} & \cdots & \frac{\partial f_{n+1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} & \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \end{pmatrix}$$

and so

$$\det\left(d_{(x_0,0)} h\right) = \det \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \neq 0.$$

By the local inversion theorem we deduce the existence of an open neighborhood of $\overline{x_0}$ in $\mathbb{R}^m$ denoted W and a neighborhood $W' \subset \mathbb{R}^m$ of $h(\overline{x_0})$ such that

$$h : W \longrightarrow W'$$

is a $\mathcal{C}^1$−diffeomorphism. Denote $g = h^{-1} : W' \longrightarrow W$ then

$$\begin{aligned} g^{-1}(x, 0) &= h(x, 0) \\ &= f(x) \end{aligned}$$

so

$$g \circ f(x) = (x, 0)$$

$$\forall x \in V = f^{-1}(W').$$

■

Exercise 3.3. *Let*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (x, e^x) . \end{aligned}$$

1. Define a $\mathcal{C}^1-$diffeomorphism $g : V \subseteq \mathbb{R}^2 \longrightarrow W \subseteq \mathbb{R}^2$ such that $g \circ f$ is a canonical injection.

Solution. *Let $x \in \mathbb{R}$. We see that*

$$d_x f = \begin{pmatrix} 1 \\ e^x \end{pmatrix}$$

and that f is an immersion with $rank_f = 1$. Define

$$\begin{aligned} h : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, e^x + y) \end{aligned}$$

and

$$d_{(x,y)} h = \begin{pmatrix} 1 & 0 \\ e^x & 1 \end{pmatrix} \implies \det \left(d_{(x,y)} h \right) = 1 \neq 0.$$

We find the inverse

$$\begin{aligned} (x, e^x + y) &= (\alpha, \beta) \\ &\implies \begin{cases} x &= \alpha \\ y &= \beta - e^\alpha \end{cases} \end{aligned}$$

so

$$\begin{aligned} g(x, y) &= h^{-1}(x, y) \\ &= (x, y - e^x) \end{aligned}$$

which is an injective map. So h is a $\mathcal{C}^1-$diffeomorphism and we have

$$\begin{aligned} g \circ f(x) &= g[x, e^x] \\ &= (x, e^x - e^x) \\ &= (x, 0) \end{aligned}$$

which is a canonical injection.

Remark 3.4. *In the previous exercise, the function g is not unique. Indeed if we set*

$$h(x, y) = (x, e^x + yk(x))$$

with k a non-zero function. We would have

$$d_{(x,y)}h = \begin{pmatrix} 1 & 0 \\ e^x + yk'(x) & k(x) \end{pmatrix}$$

and

$$\det\left(d_{(x,y)}h\right) = k(x) \neq 0.$$

Set $g = h^{-1}$ and we find that

$$g(x,y) = \left(x, \frac{y - e^x}{k(x)}\right)$$

and that

$$\begin{aligned} g \circ f(x) &= g[x, e^x] \\ &= \left(x, \frac{e^x - e^x}{k(x)}\right) \\ &= (x, 0). \end{aligned}$$

§3.4 Constant rank theorem

This theorem states that a function whose differential maintains the same rank r in a neighborhood of a certain point can be viewed as a projection onto the first r coordinates.

Theorem 3.4.1

Let U be an open subset of $\mathbb{R}^n$, a map of class $\mathcal{C}^1$ $f : U \longrightarrow \mathbb{R}^m$ and an integer $r \leq \min(n, m)$. If $\operatorname{rank}_{x_0} f = r, \forall x_0 \in U$ then $\forall x \in U$ there exists

1. an open neighborhood $V \subset U$ of x,
2. an open neighborhood $W \subset \mathbb{R}^m$ of $f(x)$,
3. a diffeomorphism $\varphi : V \longrightarrow \varphi(V) \subset \mathbb{R}^n$,
4. and a diffeomorphism $\psi : W \longrightarrow \psi(W) \subset \mathbb{R}^m$,

such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \downarrow & & \downarrow \psi \\ \varphi(V) & \underset{\psi \circ f \circ \varphi^{-1}}{\longrightarrow} & \psi(W) \end{array}$$

and

$$\psi \circ f \circ \varphi^{-1}(y_1, \cdots, y_r, y_{r+1}, \cdots, y_m) = (y_1, \cdots, y_r, 0, \cdots, 0).$$

Proof. At point $x = (x_1, \cdots, x_r, x_{r+1}, \cdots, x_n)$ for the function

$$f(x) = (f_1(x), \cdots, f_r(x), f_{r+1}(x), \cdots, f_m(x))$$

we have

$$d_x f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} & \cdots & \frac{\partial f_r}{\partial x_n} \\ \frac{\partial f_{r+1}}{\partial x_1} & \cdots & \frac{\partial f_{r+1}}{\partial x_r} & \cdots & \frac{\partial f_{r+1}}{\partial x_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_r} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

and since $\text{rank}_x f = r$ we can assume that

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} \end{pmatrix} \neq 0.$$

Since the function associating to a matrix its determinant is continuous, we deduce the existence of a neighborhood of x_0 denoted $\overline{V} \subset U$ such that

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} \end{pmatrix} \neq 0, \forall x \in \overline{V}.$$

If we set

$$\begin{aligned} \varphi : \overline{V} &\longrightarrow \mathbb{R}^n \\ x = (x_1, \cdots, x_r, x_{r+1}, \cdots, x_n) &\longmapsto \varphi(f_1(x), \cdots, f_r(x), x_{r+1}, \cdots, x_n) \end{aligned}$$

then the function φ is of class $\mathcal{C}^1$ and

$$d_x\varphi = \begin{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} \end{pmatrix} & \begin{pmatrix} \frac{\partial f_1}{\partial x_{r+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_{r+1}} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \end{pmatrix}$$

while having

$$\det\left(d_x\varphi\right) = \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} \end{pmatrix} \neq 0.$$

By the local inversion theorem, there exists an open neighborhood $\tilde{V} \subset \overline{V}$ of x_0 such that the function

$$\varphi : \tilde{V} \subset \mathbb{R}^n \longrightarrow \varphi\left(\tilde{V}\right) \subset \mathbb{R}^n$$

is a $\mathcal{C}^1-$diffeomorphism. Note that if

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_r \\ y_{r+1} \\ \vdots \\ y_n \end{pmatrix} = \varphi\left(x\right) = \begin{pmatrix} f_1\left(x\right) \\ \vdots \\ f_r\left(x\right) \\ x_{r+1} \\ \vdots \\ x_n \end{pmatrix}$$

we have by correspondence $y_i = f_i\left(x\right), \forall i = 1, \cdots, r$. If we denote

$$h = f \circ \varphi^{-1} : \varphi\left(\tilde{V}\right) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

then for $y = \left(y_1, \cdots, y_r, y_{r+1}, \cdots, y_n\right)$ we obtain, knowing that $x = \varphi^{-1}\left(y\right)$

$$\begin{aligned} h\left(y\right) &= f \circ \varphi^{-1}\left(y\right) \\ &= \left(h_1\left(y\right), \cdots, h_r\left(y\right), h_{r+1}\left(y\right), \cdots, h_m\left(y\right)\right) \\ &= f\left(x\right) \\ &= \left(f_1\left(x\right), \cdots, f_r\left(x\right), f_{r+1}\left(x\right), \cdots, f_m\left(x\right)\right) \\ &= \left(y_1, \cdots, y_r, h_{r+1}\left(y\right), \cdots, h_m\left(y\right)\right). \end{aligned}$$

Computing the differential

$$d_y h = \begin{pmatrix} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} \frac{\partial h_{r+1}}{\partial y_1} & \cdots & \frac{\partial h_{r+1}}{\partial y_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial y_1} & \cdots & \frac{\partial h_m}{\partial y_r} \end{pmatrix} & \begin{pmatrix} \frac{\partial h_{r+1}}{\partial y_{r+1}} & \cdots & \frac{\partial h_{r+1}}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial y_{r+1}} & \cdots & \frac{\partial h_m}{\partial y_n} \end{pmatrix} \end{pmatrix}.$$

And since $\text{rank}_x f = r$ and φ is a $\mathcal{C}^1-$diffeomorphism, we deduce that $\text{rank}_y h = r$ and that $\forall y \in \varphi\left(\tilde{V}\right)$

$$\begin{pmatrix} \frac{\partial h_{r+1}}{\partial y_{r+1}} & \cdots & \frac{\partial h_{r+1}}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial y_{r+1}} & \cdots & \frac{\partial h_m}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

which means the functions $h_{r+1}, \cdots, h_m$ are independent of $y_{r+1}, \cdots, y_n$

$$h_i(y) = h_i(y_1, \cdots, y_r), r+1 \leq i \leq m.$$

Let $W \subset \mathbb{R}^m$ be an open neighborhood of $f(x_0)$ such that $x_0 \in f^{-1}(W) \subset \tilde{V}$ and the map

$$\begin{aligned} \psi : W &\longrightarrow \mathbb{R}^m \\ z = (z_1, \cdots, z_m) &\longmapsto (z_1, \cdots, z_r, z_{r+1} - h_{r+1}(z), \cdots, z_m - h_m(z)) \end{aligned}$$

so

$$d_z\psi = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{-\partial h_{r+1}}{\partial z_1} & \cdots & \frac{-\partial h_{r+1}}{\partial z_r} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\partial h_m}{\partial z_1} & \cdots & \frac{-\partial h_m}{\partial z_r} & 0 & \cdots & 1 \end{pmatrix}.$$

This map is injective and a local $\mathcal{C}^1-$diffeomorphism so by the local inversion theorem ψ is a $\mathcal{C}^1-$diffeomorphism on all W. Denoting $V = f^{-1}(W)$ then the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi \downarrow & & \downarrow \psi \\ \varphi(V) & \underset{\psi \circ f \circ \varphi^{-1}}{\longrightarrow} & \psi(W) \end{array}$$

and we have

$$\begin{aligned} \psi \circ f \circ \varphi^{-1}(y_1, \cdots, y_r, y_{r+1}, \cdots, y_m) &= \psi(y_1, \cdots, y_r, h_{r+1}(y), \cdots, h_m(y)) \\ &= (y_1, \cdots, y_r, h_{r+1}(y) - h_{r+1}(y), \cdots, h_m(y) - h_m(y)) \\ &= (y_1, \cdots, y_r, 0, \cdots, 0). \end{aligned}$$

■

Remark 3.5. *In the special case where f is a submersion (resp. immersion) we recover the submersion (resp. immersion) characterization theorem.*

§3.5 Exercices

Exercice 3.1. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $f(x,y) = (x^2 - y, x^2 + y^2)$ *and* $g = f \circ f$.

1. *Show that* f *and* g *are of class* C^1.
2. *Compute at every point* $(x,y) \in \mathbb{R}^2$ *the Jacobian matrix of* f *denoted* $D_{(x,y)}f$*; compute the Jacobian matrix of* g *at the point* $(0,0)$ *denoted* $D_{(0,0)}g$.
3. *Show that there exists* $\rho > 0$ *such that for all* $(x,y) \in \overline{B_\rho((0,0))}$ *(the closed ball centered at* $(0,0)$ *with radius* ρ*) we have* $||D_{(x,y)}g|| \leq \frac{1}{2}$.
4. *Show that the function* g *has a unique fixed point in* $\overline{B_\rho((0,0))}$.

Exercice 3.2. *Consider the map* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $F(x,y) = (\cos x - \sin y,\ \sin x - \cos y)$*; denote by* $F^{(k)}$ *the* k*-fold composition of* F.

1. *Show that* $||F'(x,y)|| \leq \sqrt{2}$ *for all* (x,y).
2. *Deduce that the recursively defined sequence given by* x_0, y_0 *and for* $n \geq 1$

$$x_{n+1} = \frac{1}{2}(\cos x_n - \sin y_n), \quad y_{n+1} = \frac{1}{2}(\sin x_n - \cos y_n)$$

converges for all (x_0, y_0). *What is its limit?*

Exercice 3.3. *Let* f *be a differentiable function from* $]a,b[\subset \mathbb{R}$ *to* $\mathbb{R}^n$*; suppose there exists* $k > 0$ *such that*

$$||f'(x)|| \leq k||f(x)||, \quad \forall x \in]a,b[.$$

Show that if f *vanishes at a point* $x_0 \in]a,b[$*, then* f *is identically zero on* $]a,b[$ *(show that* $E = \{x \in]a,b[\ ;\ f(x) = 0\}$ *is open).*

Exercice 3.4. *Let* E, F *be normed spaces,* Ω *an open subset of* E *and* $f : \Omega \to F$ *a continuous map.*

1. *Let* a *be a point of* Ω. *If* f *is differentiable in* $\Omega \setminus \{a\}$ *and if the map* $x \in \Omega \setminus \{a\} \mapsto Df(x)$ *has a limit* $T \in \mathcal{L}(E,F)$ *when* x *tends to* a *in* Ω*, show that* f *is differentiable at the point* a *and that* $Df(a) = T$ *(apply the mean value theorem to the function* $g : x \mapsto f(x) - T(x)$*).*
2. *Suppose* f *is differentiable in* Ω. *Show that* $Df : \Omega \to \mathcal{L}(E,F)$ *is continuous at* $a \in \Omega$ *if and only if, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\|f(a+h) - f(a+k) - Df(a)(h-k)\| \leq \epsilon\|h-k\| \quad \text{if} \quad \|h\| < \delta \text{ and } \|k\| < \delta.$$

3. *Suppose now that there exists a continuous map* $x \in \Omega \mapsto T_x \in \mathcal{L}(E, F)$ *such that for every* $x \in \Omega$ *and every* $h \in E$
$$\lim_{t \to 0, t \neq 0} \frac{f(x+th) - f(x)}{t} = T_x(h).$$
Show that f *is of class* $\mathcal{C}^1$ *and that* $Df(x) = T_x$ *for every* $x \in \Omega$. *(Consider the function* $g(t) = f(x+th) - tT_x(h)$*.)*

Exercice 3.5. *Show that if* $f : E \to F$ *is a* $\mathcal{C}^1$*-diffeomorphism then:*

1. $\forall a \in E, d_a f \in$ *Isom*(E, F).
2. $\dim E = \dim F$.

Exercice 3.6. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a function of class* $\mathcal{C}^1$ *on* $\mathbb{R}$ *such that there exists* $k > 0$ *with* $\forall t \in \mathbb{R}$ *we have* $|g'(t)| \leq k$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *with* $f(x, y) = (x + g(y), y + g(x))$.

1. *Show that* f *is of class* $\mathcal{C}^1$ *on* $\mathbb{R}^2$ *and determine its differential.*
2. *Show that* f *is injective and deduce that it is bijective onto its image* $f(\mathbb{R}^2)$.
3. *Show that* f^{-1} *is differentiable on* $f(\mathbb{R}^2)$.

Exercice 3.7. *Does the relation*
$$xe^y + ye^x = 1$$
define a function $y = \varphi(x)$ *in a neighborhood of* 0 *with value* 1 *at* 0*?*

Exercice 3.8. *Consider* $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $f(x, y, z) = x^2 - xy^3 - y^2 z + z^3$ *and the surface* S *with equation* $f(x, y, z) = 0$.

1. *Show that in a neighborhood of the point* $(1, 1, 1)$, *the surface* S *is defined by an equation of the type* $z = \phi(x, y)$ *where* ϕ *is a function of class* $\mathcal{C}^\infty$ *defined in a neighborhood of* $(1, 1)$.
2. *Determine the equation of the tangent plane at* $(1, 1, 1)$ *to* S.
3. *Compute the second order partial derivatives of* ϕ *in a neighborhood of the point* $(1, 1)$.
4. *What is the position of* S *relative to its tangent plane?*

Exercice 3.9. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $f(x, y) = (e^x \cos y, e^x \sin y)$.

1. *Show that* f *defines a surjective map from* $\mathbb{R}^2$ *onto* $\mathbb{R}^2 \setminus \{(0, 0)\}$.
2. *Let* $(x_0, y_0) \in \mathbb{R}^2$.
   (a) *Compute the Jacobian matrix of* f *at* (x_0, y_0) *and deduce that* f *defines a local* $\mathcal{C}^1$*-diffeomorphism in a neighborhood of* (x_0, y_0).
   (b) *Does* f *realize a* $\mathcal{C}^1$*-diffeomorphism from* $\mathbb{R}^2$ *onto* $\mathbb{R}^2 \setminus \{(0, 0)\}$*?*

Chapter 4

Submanifolds

Definition 4.0.1

Let E be a $\mathbb{R}-$vector space of dimension n and $V \subset E$ non-empty. We say that V is a submanifold of class $\mathcal{C}^k$ and dimension $m \leq n$ if $\forall x_0 \in V$ there exists

1. an open neighborhood U of x_0,
2. an open neighborhood U' of 0 in $\mathbb{R}^m$,
3. a $\mathcal{C}^k-$diffeomorphism

$$\begin{aligned} \phi : U &\longrightarrow U' \\ x &\longmapsto (\phi_1(x), \phi_2(x), \cdots, \phi_m(x)) \end{aligned}$$

such that:

$$\phi(U \cap V) = (\mathbb{R}^m \times \{0\}) \cap U'$$

in other words if

$$x \in U \cap V \implies \phi_{m+1}(x) = \cdots = \phi_n(x) = 0.$$

We call ϕ a local straightening of V onto $\mathbb{R}^m \times \{0\}$.

Remark 4.1.

- *If $k = \infty$ we say the submanifold V is smooth.*
- *We call the codimension of V the number $codim(V) = n - \dim(V)$.*
- *If*

$$\begin{cases} \dim(V) = 1 & V \text{ is a curve} \\ \dim(V) = 2 & V \text{ is a surface} \\ codim(V) = 1 & V \text{ is a hyperplane.} \end{cases}$$

Example 4.1. *Let $f : U \subset \mathbb{R}^p \longrightarrow \mathbb{R}^m$ be a function of class $\mathcal{C}^k$. Show that its graph*

$$\Gamma_f = \{(x, f(x)), x \in U\} \subset U \times \mathbb{R}^m$$

is a submanifold of $\mathbb{R}^{p+m}$.

Solution. *Define the function*

$$\begin{aligned}\phi : U \times \mathbb{R}^m &\longrightarrow \mathbb{R}^{m+p}\\ (x,y) &\longmapsto (x, y - f(x))\end{aligned}$$

and show that it is a $\mathcal{C}^k-$*diffeomorphism.*

1. ϕ is bijective.

   - Injection: Let (x_1, y_1) and $(x_2, y_2) \in U \times \mathbb{R}^m$ such that

     $$\begin{aligned}\phi(x_1, y_1) &= \phi(x_2, y_2)\\ &\Rightarrow \begin{cases} x_1 & = x_1\\ y_1 - f(x_1) & = y_2 - f(x_2)\end{cases}\\ &\Rightarrow \begin{cases} x_1 & = x_1\\ y_1 & = y_2\end{cases}\end{aligned}$$

     so ϕ is injective.
   - Surjection: let $(u, v) = \phi(x, y) = (x, y - f(x))$

     $$\begin{aligned}&\Rightarrow \begin{cases} u & = x\\ v & = y - f(x)\end{cases}\\ &\Rightarrow \begin{cases} x & = u\\ y & = v + f(u)\end{cases}\end{aligned}$$

     so ϕ is surjective and bijective and its inverse

     $$\begin{aligned}\phi^{-1} : \mathbb{R}^{m+p} &\longrightarrow U \times \mathbb{R}^m\\ (x,y) &\longmapsto (x, y + f(x)).\end{aligned}$$

2. Since f is of class $\mathcal{C}^k$ then ϕ and ϕ^{-1} are also, so ϕ is a $\mathcal{C}^k-$diffeomorphism.

We then have

$$\begin{aligned}\phi\left(\Gamma_f \cap \mathbb{R}^{m+p}\right) &= \phi(\Gamma_f)\\ &= \{\phi(x,y), (x,y) \in \Gamma_f\}\\ &= \{(x, y - f(x)), x \in U \text{ and } y = f(x)\}\\ &= \{(x, 0), x \in U\}\\ &= (\mathbb{R}^p \times \{0\}) \cap \mathbb{R}^{p+m}\end{aligned}$$

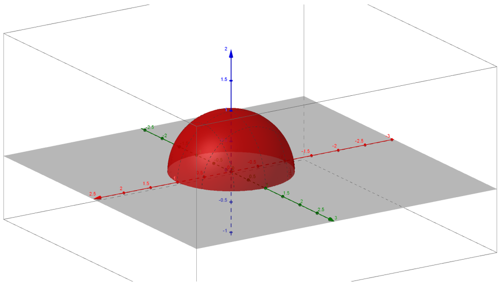
so

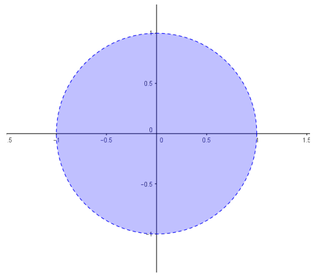
$$\phi\left(\underbrace{\Gamma_f}_{V} \cap \underbrace{\mathbb{R}^{m+p}}_{U}\right) = (\mathbb{R}^p \times \{0\}) \cap \underbrace{\mathbb{R}^{p+m}}_{U'}$$

Example 4.2. *The set*

$$V = \left\{(x;y;z) \in \mathbb{R}^3; x^2+y^2+z^2 = 1\text{‘ } \textit{and } z > 0\right\}$$

is a submanifold of dimension 2.





Indeed, consider the disk

$$D = \left\{(x,y) \in \mathbb{R}^2; x^2+y^2 < 1\right\}$$

and set $U = D \times \mathbb{R}$ *and the function*

$$\begin{aligned} f : U &\longrightarrow U \\ (x,y,z) &\longmapsto \left(x, y, z - \sqrt{1-x^2-y^2}\right). \end{aligned}$$

We have $V \subset U$ *and*

$$d_{(x,y,z)}f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x}{\sqrt{1-x^2-y^2}} & \frac{y}{\sqrt{1-x^2-y^2}} & 1 \end{pmatrix}$$

so f *is differentiable on* U. *We have*

$$\begin{aligned} f(V) &= \{f(x,y,z), (x,y,z) \in V\} \\ &= \left\{f(x,y,z), x^2+y^2+z^2 = 1\text{‘ } \textit{and } z > 0\right\} \\ &= \left\{\left(x, y, z - \sqrt{1-x^2-y^2}\right), x^2+y^2+z^2 = 1\text{‘ } \textit{and } z > 0\right\} \\ &= \{(x,y,0)\} \\ &= D \times \{0\} \\ &\equiv D. \end{aligned}$$

Exercise 4.1. *Show that the set*

$$V = \left\{\begin{pmatrix} \sin(2t)\cos(t) \\ \cos^2(t) \end{pmatrix}, \frac{-\pi}{2} < t < \frac{\pi}{3}\right\}$$

is a submanifold of dimension 1.

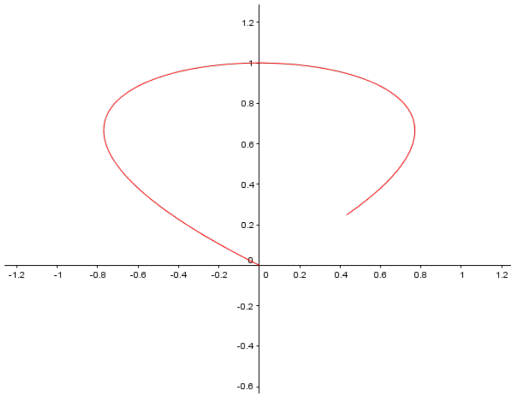


Figure 4.1: The curve V
fig:curve1

Exercise 4.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$. Then the curve*

$$\Gamma_f = \{(x, f(x)), x \in I\}$$

is a submanifold of dimension 1.

Indeed, by setting:

$$\begin{aligned} \varphi : I \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, y - f(x)) \end{aligned}$$

we find that φ is injective and that

$$d_{(x,y)}\varphi = \begin{pmatrix} 1 & 0 \\ f'(x) & 1 \end{pmatrix}$$

and $\det\left(d_{(x,y)}\varphi\right) = 1 \neq 0$ so φ is a $\mathcal{C}^1-$diffeomorphism from $U = I \times \mathbb{R}$ into $\varphi(U)$ and we have

$$(x, y) \in V \Longleftrightarrow y - f(x) = 0.$$

In general, if $f : V \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}^q$ is a function of class $\mathcal{C}^1$ then the set

$$\Gamma_f = \left\{(x, f(x)) \in \mathbb{R}^{p+q}, x \in V\right\}$$

is a submanifold of dimension p. Indeed, for the injective map of class $\mathcal{C}^1$

$$\begin{aligned} \varphi : V \times \mathbb{R}^q &\longrightarrow \mathbb{R}^{p+q} \\ (x, y) &\longmapsto (x, y - f(x)) \end{aligned}$$

we have

$$d_{(x,y)}\varphi = \begin{pmatrix} \mathbf{Id}_{p\times p} & \mathbf{0} \\ -d_x f & \mathbf{Id}_{q\times q} \end{pmatrix}$$

and so it is a $\mathcal{C}^1-$diffeomorphism from $U = V \times \mathbb{R}^q$ onto $\varphi(U)$ and by correspondence

$$\begin{pmatrix} \varphi_{p+1}(x,y) \\ \varphi_{p+2}(x,y) \\ \vdots \\ \varphi_{p+q}(x,y) \end{pmatrix} = y - f(x)$$
$$= 0.$$

§4.1 Submanifolds defined by equations

§4.1.1 Regular values

Let $f : U \subset E \longrightarrow F$ be a differentiable function on its domain. We say that $c \in F$ is a regular value of f if and only if f is a submersion at every point of the set $f^{-1}\{c\}$. Or in other words, there exists an open neighborhood of $f^{-1}\{c\}$ denoted U' on which f is of class $\mathcal{C}^1$ and $d_x f$ is injective $\forall x \in U'$.

Remark 4.2. *If c is a regular value of f such that the set $f^{-1}\{c\} \neq \emptyset$, we say that this set is a regular level of f and denote it by $f = c$.*

We will present a result that is the analog of the implicit function theorem but from a geometric viewpoint. It is a theorem that will allow us to show that a subset of E is a submanifold.

Theorem 4.1.1

Let E and F be two normed vector spaces of finite dimensions. Let $f : U \subset E \longrightarrow F$ be a function of class $\mathcal{C}^k$ with $k \in \mathbb{N} \cup \{\infty\}$. Then every regular level of f is a submanifold of class $\mathcal{C}^k$ of E and of codimension equal to $\dim F$.

Remark 4.3.

1. *If U is a submanifold then* $\dim E = \dim E - \dim F$.
2. *If f is a submersion then its differential df is surjective and* $\mathrm{rank} f = \dim F$.

Exercise 4.3. *Show that the unit sphere in* $\mathbb{R}^3$

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a submanifold of dimension 2.

Solution. *Consider the function* $f : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$. *Then*

$$\begin{aligned} S &= \left\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = 0\right\} \\ &= f^{-1}\{0\}. \end{aligned}$$

We then have

$$d_{(x,y,z)}f = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which is surjective except at the origin which does not belong to S *and so* $rank_{(x,y,z)}f = 1 = \dim \mathbb{R}$ *which is the target space of* f *so the function* f *is a submersion on* $f^{-1}\{0\}$ *and so* S *is a submanifold of dimension equal to* $\dim \mathbb{R}^3 - \dim \mathbb{R} = 2$.

§4.2 Exercices

Exercice 4.1. *Are the following functions immersions or submersions?*

1. $f : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (x, y, 0)$.
2. $f : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (y, z)$.
3. $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto xy + 2yz + 3xz$.
4. $f : \mathbb{R} \to \mathbb{R}^2, t \mapsto (\sin(2t), \sin(3t))$.
5. $f : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x^2 + y^2 + z^2, xy)$.
6. $f : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (e^x, \cos(y), \sin(y))$.

Exercice 4.2. *Are the following sets submanifolds? If yes, give the dimension.*

1. $\mathcal{S}_1 = \{(x, y, z) \in \mathbb{R}^3; z = x - 2(x^2 + y^2)\}$.
2. $\mathcal{S}_2 = \{(t, t^2); t \in \mathbb{R}\}$.
3. $\mathcal{S}_3 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$.
4. $\mathcal{S}_4 = \{(x, y) \in \mathbb{R}^2; xy = 0\}$.

5. $\mathcal{S}_5 = \{(x, y) \in \mathbb{R}^2; x > 0 \text{ and } y \geq 0\}$.

Exercice 4.3. *For which values of the real parameter α is the set*

$$\mathcal{C} = \Big\{(x, y) \in \mathbb{R}^2; x^2 - y^2 = \alpha\Big\}$$

a submanifold of $\mathbb{R}^2$?

Exercice 4.4. *Let M_1 be a submanifold of $\mathbb{R}^n$ of dimension p_1 and M_2 be a submanifold of $\mathbb{R}^m$ of dimension p_2. Show that*

$$M_1 \times M_2 = \Big\{a = (a_1, a_2) \in \mathbb{R}^{n+m}; a_1 \in M_1, a_2 \in M_2\Big\}$$

is a submanifold of $\mathbb{R}^{n+m}$ and specify its dimension.

Exercice 4.5. *Show that the torus of revolution T^2, obtained by rotating a circle of radius r around a line in its plane located at a distance $d > r$ from its center, is a submanifold of $\mathbb{R}^3$.*

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