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Département de Mathématiques

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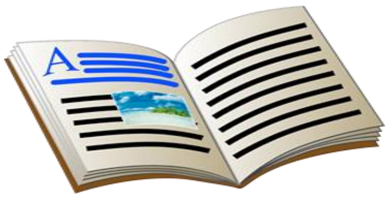

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Functional Analysis

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Chapter 1

Introduction

These lecture notes are designed for first-year Master's students in Operational Research (first semester). They provide an introduction to the fundamental concepts and theorems of functional analysis, a branch of mathematics that studies infinite-dimensional vector spaces with topological structures.

Functional analysis is an essential tool for modern applied mathematics, particularly in optimisation, control theory, approximation theory, and the analysis of partial differential equations. For students in Operational Research, a solid grasp of functional analysis is indispensable when dealing with optimisation problems in infinite-dimensional spaces, variational inequalities, or the study of operators arising in models of operations research.

The course begins with a brief review of general topology (topological spaces, continuity, compactness, sequences) and then focuses on metric spaces, where the notions of completeness and compactness are studied in detail. Normed vector spaces and Banach spaces are introduced as the natural framework for linear analysis. The concepts of connectedness and convexity are treated both topologically and algebraically, with an emphasis on their role in optimisation.

The central part of the notes presents the three great theorems of functional analysis: the Banach–Steinhaus theorem (uniform boundedness), the open mapping theorem, and the closed graph theorem. All three rely on Baire's category theorem and are cornerstones of the theory of linear operators between Banach spaces. The final chapter is devoted to Hilbert spaces, including orthogonal projections, the Riesz–Fréchet representation theorem, the Lax–Milgram theorem, Hilbert bases, and weak convergence. Applications to L^2 spaces are given as examples.

Each chapter is accompanied by numerous exercises, ranging from routine verifications to more challenging problems, aimed at helping the student master the concepts and develop the necessary skills for further study in functional analysis and its applications to operational research.

Prerequisites: Basic knowledge of linear algebra, calculus, and elementary topology (open and closed sets, continuity, compactness in $\mathbb{R}^n$) is assumed. Familiarity with Lebesgue integration is useful for the last chapter, but the essential facts are recalled when needed.

Structure of the notes:

1. A Little General Topology
2. Metric Spaces

3. Normed Vector Spaces
4. Connectedness and Convexity
5. The Great Theorems of Functional Analysis (Baire, Banach–Steinhaus, Open Mapping, Closed Graph)
6. Hilbert Spaces

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H. Banouh
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Chapter 2

A Little General Topology

§2.1 Definitions

[1–11]Formalize the ”intuitive” concepts of proximity between mathematical objects of the ”same nature”, and continuity of maps acting on these objects.

Definition 2.1.1

Let X be a set. A **topology** on X is a set $\mathcal{O}$ of subsets of X satisfying the following properties:

1. $\mathcal{O}$ contains $\emptyset$ and X,
2. arbitrary unions of elements of $\mathcal{O}$ are in $\mathcal{O}$,
3. finite intersections of elements of $\mathcal{O}$ are in $\mathcal{O}$.

The elements of $\mathcal{O}$ are called **open sets** and their complements are called **closed sets**. The pair $(X, \mathcal{O})$ is called a **topological space**.

Example 2.1.

1. *Let X be a set. Then $\{\emptyset, X\}$ is a topology on X, sometimes called the **trivial topology** or **indiscrete topology**.*
2. *Taking $\mathcal{O}$ to be the set of all subsets of X also gives a topology, called the **discrete topology**.*
3. *On $\mathbb{R}$, the set of arbitrary unions of open intervals $[a, b]$ is a topology. Unless explicitly stated otherwise, we always equip $\mathbb{R}$ with this topology.*

Definition 2.1.2

Let X be a set and $\mathcal{O}_1$, $\mathcal{O}_2$ two topologies on X. We say that $\mathcal{O}_1$ is **finer** (or stronger) than $\mathcal{O}_2$ if $\mathcal{O}_2 \subset \mathcal{O}_1$.

Thus, the discrete topology is the finest and the trivial topology the coarsest of all topologies. In the case of $\mathbb{R}$, the ”usual” topology lies between these two.

The data of a topology $\mathcal{O}$ on the set X allows us to define the notions of neighborhood, closure, interior, boundary, etc.

Definition 2.1.3

Let $(X, \mathcal{O})$ be a topological space.

- Let $x \in X$ and $V \subset X$. We say that V is a **neighborhood** of x if V contains an open set containing x. The set of neighborhoods of x is denoted $\mathcal{V}_X(x)$.
- A subset $\mathcal{V}$ of $\mathcal{V}_X(x)$ is called a **neighborhood base** of x if every element of $\mathcal{V}_X(x)$ contains an element of $\mathcal{V}$.
- We say that x is an **interior point** of V if V is a neighborhood of x.
- The **closure** $\bar{A}$ of a subset A of X is the smallest closed set containing A. We say that A is **dense** in X if $\bar{A} = X$.
- The **interior** $\hat{A}$ of a subset A of X is the largest open set contained in A.
- The set $\bar{A} \backslash \hat{A}$ is called the **boundary** of A.

Example 2.2. *The set of open sets containing a given point x is a neighborhood base of x.*

Remark 2.1. *The definition of a ball (seen in undergraduate studies in the context of normed vector spaces) has no equivalent in a "general" topological space.*

Exercise 2.1. *Let $(X, \mathcal{O})$ be a topological space and A a subset of X. Show the following properties:*

1. *$A = \bar{A}$ if and only if A is closed.*
2. *$A = \hat{A}$ if and only if A is open.*
3. *A is open if and only if A is a neighborhood of all its points.*
4. *$X \backslash \bar{A} = \widehat{X \backslash A}$ and $X \backslash \hat{A} = \overline{X \backslash A}$.*
5. *The boundary of A is equal to $X \backslash (\hat{A} \cup \widehat{X \backslash A})$.*

Proposition 2.1.1

Let A be a subset of a topological space $(X, \mathcal{O})$.

1. A point x of X is in $\bar{A}$ if and only if every neighborhood of x intersects A.
2. The set A is dense in X if and only if A intersects every non-empty open set of X.

Proof. 1. If $x \notin \bar{A}$ then x belongs to the open set $X \backslash \bar{A}$ which is a neighborhood of x that does not intersect A.

Conversely, if there exists an open set Ω containing x and disjoint from A, then $X \backslash \Omega$ is a closed set containing A, hence containing $\bar{A}$. Consequently, x is not in $\bar{A}$.

2. Let A be dense, Ω a non-empty open set of X, and x a point of Ω. Then Ω is a neighborhood of $x \in \bar{A}$, so it intersects A. Conversely, assume that every non-empty open set of X intersects A. Let $x \in X$ be arbitrary and V a neighborhood of x (which we can always assume to be open). Then V intersects A, so $x \in \bar{A}$, hence $\bar{A} = X$.

■

Definition 2.1.4

A topological space $(X, \mathcal{O})$ is called **Hausdorff** if for every pair (x, y) of distinct points of X there exists a neighborhood V of x and a neighborhood V' of y such that $V \cap V' = \emptyset$.

Example 2.3. *1. Let X be a set containing at least two elements and equipped with the trivial topology $\mathcal{O} = \{\emptyset, X\}$. Then $(X, \mathcal{O})$ is not Hausdorff.*

2. A set equipped with the discrete topology is always Hausdorff. Indeed, for this topology, every singleton is a neighborhood of the point it contains.

3. The set $\mathbb{R}$ equipped with its usual topology is Hausdorff.

Proposition 2.1.2

Let $(X, \mathcal{O})$ be a topological space and $A \subset X$. The set $\mathcal{O}_A$ of subsets Ω of A such that there exists $\Omega' \in \mathcal{O}$ satisfying $\Omega = \Omega' \cap A$ is a topology on A. It is called the **induced topology** by $\mathcal{O}$ on A.

§2.2 Continuity

Remark 2.2. *Unless stated otherwise, we always equip subsets of a topological set with the induced topology.*

Two simple cases to remember: if A is an open set of X then every open set of A is an open set of X; if A is a closed set of X then every closed set of A is a closed set of X.

It is now time to address the second part of the goal: generalizing the notion of continuity to functions defined on topological spaces. Without a distance or norm, we can no longer use a definition with ε and η. The use of neighborhoods is the right substitute. Let's start by defining the notion of limit.

Definition 2.2.1

Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be two topological spaces. Let A be a subset of X, $f \in F(A; X')$, $x_0 \in \bar{A}$ and $\ell \in X'$. We say that f has limit ℓ at x_0 if

$$\forall V' \in \mathcal{V}_{X'}(\ell), \exists V \in \mathcal{V}_X(x_0), (x \in A \cap V \Rightarrow f(x) \in V').$$

Remark 2.3. *In the above definition, we can of course replace $\mathcal{V}_{X'}(\ell)$ by a neighborhood base of ℓ.*

Exercise 2.2. *Show the uniqueness of the limit in the case where $(X', \mathcal{O}')$ is Hausdorff. What can be said in the general case?*

We can now define continuity:

Definition 2.2.2

Under the previous hypotheses, if moreover $x_0 \in A$ and $f(x_0) = \ell$, we say that f is continuous at x_0. If f is continuous at every point of A, we say that f is continuous on A.

To show continuity at every point, there is no need to resort to the definition thanks to the following theorem.

Theorem 2.2.1

Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be two topological spaces and $f \in F(X; X')$. The following three properties are equivalent:

1. the function f is continuous on X,
2. the preimage by f of every open set of X' for the topology $\mathcal{O}'$ is an open set of X for the topology $\mathcal{O}$,
3. the preimage by f of every closed set of X' for the topology $\mathcal{O}'$ is a closed set of X for the topology $\mathcal{O}$.

Proof. Let's just show the equivalence between the first two assertions.

Let f be continuous on X and Ω' an open set of X'. Consider $x \in f^{-1}(\Omega')$ and set $x' = f(x)$. The open set Ω' is a neighborhood of x'. By continuity of f at x, there exists a neighborhood V of x such that $f(V) \subset \Omega'$. This means that $V \subset f^{-1}(\Omega')$. So $f^{-1}(\Omega')$ is indeed open.

Conversely, assume (ii) is satisfied, and fix $x \in X$ and V' a neighborhood of $f(x)$ (which we can always assume to be open by shrinking it if necessary). Then $f^{-1}(V')$ contains x and is open thanks to (ii). It is therefore a neighborhood of x whose image is contained in V'. This ensures continuity at x. ∎

Theorem 2.2.2: Composition

Let $(X, \mathcal{O})$, $(X', \mathcal{O}')$ and $(X'', \mathcal{O}'')$ be three topological spaces. The composition of a continuous function $f : X \to X'$ and a continuous function $f' : X' \to X''$ is a continuous function $f' \circ f : X \to X''$.

Proof. Thanks to the previous theorem, the proof is very easy. Indeed, if Ω'' is an open set of X'', then, by continuity of f', the set $f'^{-1}(\Omega'')$ is an open set of X', and then, by continuity of f, the set $f^{-1}(f'^{-1}(\Omega''))$ is an open set of X. It remains to note that $(f' \circ f)^{-1}(\Omega'') = f^{-1}(f'^{-1}(\Omega''))$. ■

§2.3 Sequences in Topological Spaces

It is often convenient to be able to express notions of abstract topology using sequences. For this, we must first give meaning to the notion of *convergence*.

Definition 2.3.1

Let $(X, \mathcal{O})$ be a topological space, $\ell \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of X. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to ℓ if for every neighborhood V of ℓ there exists $N \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N$.

Exercise 2.3. *In the case of a Hausdorff topological space, show the uniqueness of the limit of a sequence. What can be said for a non-Hausdorff topological space?*

Proposition 2.3.1

Let F be a closed set of $(X, \mathcal{O})$ and $(x_n)_{n\in\mathbb{N}}$ a convergent sequence of points of F. Then the limit of this sequence is also in F. We say that F is **sequentially closed**.

Proof. Assume by contradiction that the sequence $(x_n)_{n\in\mathbb{N}}$ admits a limit x in the complement Ω of F. Since Ω is a neighborhood (open) of x, the terms of the sequence must all lie in Ω beyond some rank, which contradicts the hypotheses. ■

Remark 2.4. *We will see later that the converse is true in a metric space (or more generally in any topological space admitting countable neighborhood bases at each point).*

Proposition 2.3.2

Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be two topological spaces, $x \in X$ and $f : X \to X'$ continuous at x. Then for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X converging to x, we have

$$\lim_{n\to+\infty} f(x_n) = f(x).$$

The property exhibited above is called **sequential continuity**. According to Proposition 2, sequential continuity is therefore a property *weaker* than continuity. There exist topological spaces for which the notion of sequential continuity is strictly weaker than that of continuity.

Exercise 2.4. *Give an example of topological spaces $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ and a function $f : X \to X'$ that is sequentially continuous but not continuous.*

Let us conclude these generalities with the definition of *cluster point* (to be compared with that of limit).

Definition 2.3.2

Let $(X, \mathcal{O})$ be a topological space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of X. We say that $a \in X$ is a **cluster point** of $(x_n)_{n\in\mathbb{N}}$ if for every neighborhood V of a and for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n \in V$.

Remark 2.5. *If there exists a **subsequence** φ such that the sequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ (called a **subsequence** of $(x_n)_{n\in\mathbb{N}}$) converges to a, then a is a cluster point of $(x_n)_{n\in\mathbb{N}}$.*

Exercise 2.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the topological space $(X, \mathcal{O})$ and A the set of cluster points of $(x_n)_{n\in\mathbb{N}}$. Denote $S_n = \{x_p / p \geq n\}$. Show that*

$$A = \bigcap_{n\in\mathbb{N}} \overline{S}_n. \tag{1.1}$$

§2.4 Compactness

Definition 2.4.1

A topological space X is **compact** if it is Hausdorff and if every open cover of X admits a finite subcover.

A subset A of an arbitrary topological space X is **compact** if, equipped with the induced topology, it is a compact topological space.

Remark 2.6. *We leave it to the reader to verify that a Hausdorff topological set is compact if and only if from every family of closed sets with empty intersection, one can extract a finite subfamily with empty intersection.*

Let us give a characterization of compact subsets that is more manageable than the definition.

Proposition 2.4.1

Let $(X, \mathcal{O})$ be a Hausdorff topological space and A a subset of X. Then A is compact if and only if from every family of open sets of X covering A, one can extract a finite subcover.

Proof. Assume first that A is compact and consider a cover $(\Omega_i)_{i\in I}$ of A by open sets of X. We then also have

$$A \subset \bigcup_{i\in I}(\Omega_i \cap A).$$

Since each $\Omega_i \cap A$ is open for the induced topology, we can extract a finite subcover $(\Omega_{i_k} \cap A)_{1\le k\le n}$ of A. We obviously have

$$A \subset \bigcup_{k=1}^{n}(\Omega_{i_k} \cap A) \subset \bigcup_{k=1}^{n}\Omega_{i_k}.$$

Conversely, assume that from every cover of A by open sets of X one can extract a finite subcover. Let $(\Omega_i')_{i\in I}$ be a cover of A by open sets of A. By definition of the induced topology, there exist open sets Ω_i of X such that $\Omega_i' = \Omega_i \cap A$. The family $(\Omega_i)_{i\in I}$ is a cover of A by open sets of X, so we can extract a finite subcover $(\Omega_{i_k})_{1\le k\le n}$. Then $(\Omega_{i_k}')_{1\le k\le n}$ is a finite subcover of A. ■

Theorem 2.4.1

Let $(X, \mathcal{O})$ be a Hausdorff topological space and A a subset of X.

1. If A is compact then A is closed.
2. Conversely, if A is closed and if moreover X is compact then A is compact.

Proof. To establish the first property, we will show that $X \setminus A$ is open. So let $x \in X \setminus A$. Since $x \notin A$ and the topology is Hausdorff, for every $a \in A$ there exist two disjoint open sets V_a and W_a such that

$$x \in V_a, a \in W_a \text{ and } V_a \cap W_a = \emptyset.$$

The family $(W_a)_{a\in A}$ constitutes a cover of the compact A by open sets, from which we can extract a finite subcover $(W_{a_i})_{1\le i\le n}$. It is then clear that $\bigcap_{i=1}^{n} V_{a_i}$ is a neighborhood of x that does not intersect A. The point x being arbitrary in $X \setminus A$, we can conclude that $X \setminus A$ is open.

To establish property (ii), consider a family $(F_i)_{i\in I}$ of closed sets of A with empty intersection. Since A is closed, each F_i is also a closed set of the compact X. By Remark 1.5, we deduce that the family of closed sets considered admits a finite subfamily with empty intersection. Consequently, the subset A is compact. ■

Proposition 2.4.2

In a compact space, every sequence has at least one cluster point. Moreover, if this cluster point is unique then the sequence converges.

Lemma 2.4.1

Let $(X, \mathcal{O})$ be a compact topological space and $(F_n)_{n\in\mathbb{N}}$ a decreasing sequence (with respect to inclusion) of non-empty closed sets of X. Then $\bigcap_{n\in\mathbb{N}} F_n$ is not empty.

Returning to the proof of the proposition. Let X be a compact space and $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$. For $n \in \mathbb{N}$, denote $S_n = \{x_p/p \geq n\}$ and A the set of cluster points of the sequence. The family $(\overline{S_n})_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty closed sets of the compact X, so, by virtue of the lemma above, its intersection (which is precisely A by (1.1)) is non-empty.

Now assume that the set A is reduced to a single point ℓ but (by contradiction) that the sequence does not converge. Then there exists a neighborhood V of ℓ which we can assume to be open, and a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ such that $x_{\varphi(n)}$ is never in V. The set $X \setminus V$ is a closed set of the compact X, hence is compact. We deduce that the considered subsequence admits a cluster point in $X \setminus V$, necessarily distinct from ℓ. This contradicts the hypothesis.

Proposition 2.4.3

Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be two topological spaces, the first being compact and the second, Hausdorff. Then the image of X by any continuous map from X to X' is compact.

Proof. Let $f : X \to X'$ be continuous and $(\Omega_i)_{i\in I}$ an open cover of $f(X)$ by open sets of X'. One easily verifies that $(f^{-1}(\Omega_i))_{i\in I}$ is a cover of X. Moreover, by continuity of f, each set $f^{-1}(\Omega_i)$ is open. We can therefore find a finite number of indices $i_1, \cdots, i_N$ such that $X \subset \bigcup_{k=1}^{N} f^{-1}(\Omega_{i_k})$, whence we deduce $f(X) \subset \bigcup_{k=1}^{N} \Omega_{i_k}$. Since X' is Hausdorff, this ensures the compactness of $f(X)$. ■

Remark 2.7. *In other words, in Hausdorff topological spaces, the image of a compact set by a continuous map is compact.*

§2.5 Exercices

Exercice 1. *Let $\lambda > 0$. For all integer $n \geq 1$, we note B_n the disc*

$$B_n = \left\{(x, y) \in \mathbb{R}^2; \left(x - \frac{1}{n}\right)^2 + \left(y - \frac{1}{n}\right)^2 \leq \frac{\lambda^2}{n^2}\right\}.$$

1. *For what condition on λ we have $B_{n+1} \subset B_n$?*

2. *Let $B = \bigcup_{n\geq 1} B_n$. Give a necessary and sufficient condition on λ for B to be closed.*

Exercice 2. *To determine the interior and closure of the following subsets of $\mathbb{R}^2$*

$$\begin{aligned} A &= \Big\{(x,y) \in \mathbb{R}^2; x > 0\Big\} \\ B &= \Big\{(x,y) \in \mathbb{R}^2; xy = 1\Big\} \\ C &= \Big\{(x,y) \in \mathbb{R}^2; xy > 1\Big\} \\ D &= \Big\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\Big\} \setminus \Big\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1\Big\}. \end{aligned}$$

Exercice 3. *Let A, B be two subsets of a vector normed space $(E, \|\cdot\|)$.*

1. *We assume that $A \subset B$. Prove that $\mathring{A} \subset \mathring{B}$ and $\bar{A} \subset \bar{B}$.*

2. *Demonstrate that $(A \overset{\circ}{\cap} B) = \mathring{A} \cap \mathring{B}$ and that $\mathring{A} \cup \mathring{B} \subset (A \overset{\circ}{\cup} B)$, but the inclusion can be strict.*

3. *Compare $\overline{A \cap B}$ and $\bar{A} \cap \bar{B}$, then $\overline{A \cup B}$ and $\bar{A} \cup \bar{B}$.*

Exercice 4.

1. *Prove that the map $d(u,v) = \frac{|u-v|}{1+|u-v|}$ define a distance over $\mathbb{R}$.*

2. *Let $X =]0, +\infty[$. For $x, y \in X$, we note*

$$\delta(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

   (a) *Demonstrate that δ is a distance over X.*
   (b) *Determine $B(1,1)$ for this distance.*
   (c) *Is $A =]\mathbf{0}, 1]$ bonded for this distance? closed?*
   (d) *Determine the open balls for this distance.*

Exercice 5. *Is the function $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto xy$ uniformly continuous?*

Chapter 3

Metric Spaces

§3.1 Definitions

Definition 3.1.1

Let X be a set. We say that a map $d : X \times X \to \mathbb{R}$ is a **distance** on X if the following three properties hold:

1. symmetry: for all $(x, y) \in X^2$, $d(x, y) = d(y, x)$,
2. positivity: for all $(x, y) \in X^2$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
3. triangle inequality: for all $(x, y, z) \in X^3$, $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is then called a **metric space**.

From the definition of a distance, we can deduce the **reverse triangle inequality** valid for all $(x, y, z) \in X^3$:

$$|d(x, y) - d(y, z)| \leq d(x, z).$$

Example 3.1. *1. The function d defined by $d(x, y) = |x - y|$ for all $(x, y) \in \mathbb{R}^2$ is a distance on $\mathbb{R}$. This is the* ***usual distance*** *on $\mathbb{R}$.*

2. The function $\overline{d}$ defined by $\overline{d}(x, y) = |\arctan x - \arctan y|$ for all $(x, y) \in \mathbb{R}^2$ is a distance on the extended real line $\overline{\mathbb{R}}$ (i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$).

3. Any non-empty set X can be equipped with the ***trivial distance*** *δ defined for all $(x, y) \in X^2$ by*

$$\delta(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Definition 3.1.2

A subset A of a metric space (X, d) is said to be **bounded** if there exists $x \in X$ and a positive real M such that

$$\forall a \in A,\ d(a, x) \leq M.$$

Remark 3.1. *One easily establishes that A is a bounded subset if and only if the supremum $\delta(A)$ of the set $\{d(x,y)/(x,y) \in A^2\}$ is finite. We say that $\delta(A)$ is the* ***diameter*** *of A.*

Example 3.2.

1. *The bounded subsets of $\mathbb{R}$ equipped with the distance d defined above are the "usual" bounded subsets (e.g., $[a,b]$, $[-3,1] \cup [1,1000000]$, etc.).*
2. *All subsets of $\mathbb{R}$ equipped with the distance $\overline{d}$ are bounded. In fact $\overline{\mathbb{R}}$ itself has diameter π.*
3. *On X equipped with the trivial distance, all subsets with at least two elements are bounded and have diameter 1. (As for singletons, they have diameter zero as in any metric space.)*

Definition 3.1.3

Suppose the set X is equipped with two distances d and d'. We say that d and d' are equivalent if there exists $C \in [1,+\infty[$ such that

$$\forall (x,y) \in X^2,\, C^{-1}d(x,y) \leq d'(x,y) \leq Cd(x,y).$$

Exercise 3.1. *Verify that if X is equipped with two equivalent distances d and d', then the bounded subsets of (X,d) are the bounded subsets of (X,d'). Deduce that on $\mathbb{R}$, the distance $d(x,y) = |\arctan x - \arctan y|$ is not equivalent to the usual distance.*

Definition 3.1.4

Let A and B be two non-empty subsets of the metric space (X,d) and $x \in X$.

- The positive real number
$$d(x,A) \stackrel{\text{def}}{=} \inf_{a \in A} d(x,a)$$
is called the **distance from x to the set A**.
- The positive real number
$$d(A,B) \stackrel{\text{def}}{=} \inf\{d(a,b)/(a,b) \in A \times B\}$$
is called the **distance from A to B**.

Proposition 3.1.1

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let δ be the function defined for all $(x_1, x_2) \in X_1 \times X_2$ and $(x_1', x_2') \in X_1 \times X_2$ by

$$\delta((x_1, x_2), (x_1', x_2')) = \max(d_1(x_1, x_1'), d_2(x_2, x_2')).$$

Then δ is a distance on $X_1 \times X_2$ called the **product distance**.

Remark 3.2. *The above definition generalizes easily to the product of a finite number of metric spaces. Moreover, one can also define the product distance between two metric spaces by*

$$\delta((x_1, x_2), (x_1', x_2')) = d_1(x_1, x_1') + d_2(x_2, x_2'),$$
$$\delta((x_1, x_2), (x_1', x_2')) = \sqrt{d_1^2(x_1, x_1') + d_2^2(x_2, x_2')}.$$

This choice hardly matters because the different distances obtained are equivalent.

Definition 3.1.5

Let (X, d) be a metric space and Y a subset of X. The restriction of the function d to the set $Y \times Y$ is a distance on Y called the **induced distance**.

§3.2 Topology of Metric Spaces

In any metric space, the distance allows us to define balls.

Definition 3.2.1

Let (X, d) be a metric space, x_0 an element of X and r a positive real number.

- The set $B_X(x_0, r) \overset{\text{def}}{=} \{x \in X, d(x_0, x) < r\}$ (also denoted $B(x_0, r)$) is called the **open ball** with center x_0 and radius r.
- The set $\overline{B}_X(x_0, r) \overset{\text{def}}{=} \{x \in X, d(x_0, x) \leq r\}$ (also denoted $\overline{B}(x_0, r)$) is called the **closed ball** with center x_0 and radius r.
- The set $S_X(x_0, r) \overset{\text{def}}{=} \{x \in X, d(x_0, x) = r\}$ (also denoted $S(x_0, r)$) is called the **sphere** with center x_0 and radius r.

Balls are the elementary pieces that allow us to define a topology on metric spaces:

Definition 3.2.2

The **topology associated** with a metric space is the topology generated by the open balls, i.e., the smallest topology containing all open balls.

Since the above definition is not very manageable, we will remember that this topology is the set of arbitrary unions of finite intersections of open balls or, better, the following characterization, left as an exercise:

Proposition 3.2.1

A subset Ω of a metric space (X, d) is **open** if and only if for every $x \in \Omega$ there exists $r > 0$ such that $B_X(x, r) \subseteq \Omega$.

A metric space is therefore a particular topological space.

Remark 3.3. *Note that the set of open (or closed) balls with a given center x constitutes a neighborhood base of x. In fact, each point has a countable neighborhood base, for example the sequence of open balls with center x and radius $1/n$.*

Example 3.3.

1. *For any metric space, the topology associated with the trivial distance coincides with the discrete topology.*
2. *The topology on $\mathbb{R}$ associated with the usual distance is none other than the usual topology introduced in the previous chapter.*

Remark 3.4. *Henceforth, if A is a non-empty subset of a metric space (X, d), we therefore have two natural ways to equip A with a topology:*

- *we can first consider the topology $\mathcal{O}$ on X associated with the distance d, then equip A with the topology $\mathcal{O}_A$ induced by $\mathcal{O}$,*
- *we can instead equip A with the distance d_A induced by d, then consider the topology $\mathcal{O}_A$ on A associated with d_A.*

Fortunately, the two topologies obtained are the same! (Verifying this is an excellent exercise).

Let us also remember the following fact which is very useful:

Proposition 3.2.2

Two equivalent distances generate the same topology.

Proof. Suppose the set X is equipped with two metrics d and d'. Let C be a constant such that

$$\forall (x, y) \in X^2, C^{-1}d(x, y) \leq d'(x, y) \leq Cd(x, y). \tag{2.1}$$

Denote by $\mathcal{O}$ and $\mathcal{O}'$ the associated topologies. Let Ω be an open set for the topology $\mathcal{O}$ and x a point of Ω. We will show that Ω is a neighborhood of x for the topology $\mathcal{O}'$, which will ensure that $\mathcal{O} \subset \mathcal{O}'$. Since Ω is a neighborhood of x for the topology $\mathcal{O}$, there exists $r > 0$ such that every point $y \in X$ with $d(x, y) < r$ is in Ω. By virtue of (2.1), we deduce that every point $y \in X$ with $d'(x, y) < C^{-1}r$ is in Ω. Consequently, Ω is indeed a neighborhood of x for the topology $\mathcal{O}'$. The point x being arbitrary in Ω, we can then conclude that Ω is an open set of $\mathcal{O}'$, and hence that $\mathcal{O} \subset \mathcal{O}'$.

The inclusion $\mathcal{O}' \subset \mathcal{O}$ is proved in the same way. It suffices to exchange the roles of d and d'. ■

Exercise 3.2. *Show that the converse of the proposition above is false. One may show that the topologies associated with the distances d and $\overline{d}$ on $\mathbb{R}$ defined respectively by*

$$d(x, y) = |x - y| \quad \textit{and} \quad \overline{d}(x, y) = |\arctan x - \arctan y|$$

are the same even though these two distances are not equivalent.

We will see that the presence of a distance gives (X, d) many properties that general topological spaces do not have. The first property is that of separation:

Proposition 3.2.3

Every metric space is a Hausdorff topological space.

Proof. Let x and y be two distinct points of the metric space (X, d). It is clear that the open balls with centers x and y respectively, and radius $d(x, y)/2$, are disjoint. Moreover, they are neighborhoods of x and y respectively. ■

In a metric space, the property of being closed can be characterized using sequences (in a general topological space, only one implication is true, see Proposition 1 page 9):

Proposition 3.2.4

In a metric space, a set is closed if and only if it is sequentially closed.

Proof. Only the converse needs to be justified. We reason by contraposition. Let A be a non-closed subset of X, and $a \in \overline{A} \setminus A$. For every $n \in \mathbb{N}$, we choose a point x_n of the (non-empty) set $A \cap B(a, 2^{-n})$. The obtained sequence is a sequence of points of A that converges to a. But a is not in A, so A is not sequentially closed. ■

Remark 3.5. *In a metric space, the closure $\overline{A}$ of a subset A is therefore the set of limits of convergent sequences of elements of A.*

Exercise 3.3. *Let A be a non-empty subset of a metric space (X, d). Show that $x \in \overline{A}$ if and only if $d(x, A) = 0$.*

In metric spaces, we have a very convenient characterization of cluster points:

Proposition 3.2.5

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in the metric space (X, d). Then a is a cluster point of $(x_n)_{n\in\mathbb{N}}$ if and only if there exists a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ that converges to a.

Proof. Only the direct implication needs to be justified (the converse implication follows from the definition of a cluster point and remains true in any topological space). Assume therefore that a is a cluster point of the sequence. By restricting to neighborhoods of a consisting of the family of open balls $B(a, 2^{-k})$ in the definition of the cluster point, we see that

$$\forall k \in \mathbb{N}, \forall N \in \mathbb{N}, \exists n \geq N, d(a, x_n) < 2^{-k}.$$

We then construct a subsequence that converges to a by recurrence: define $\varphi(0)$ as the smallest index such that $d(a, x_{\varphi(0)}) < 1$, then $\varphi(1)$ as the smallest index strictly greater than $\varphi(0)$ satisfying $d(a, x_{\varphi(1)}) < 1/2$, etc. ■

Example 3.4. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbb{R}$ equipped with the usual distance. Recall that, by definition,*

$$\liminf x_n = \lim_{n\to+\infty} \inf_{p\geq n} x_p \quad \textit{and} \quad \limsup x_n = \lim_{n\to+\infty} \sup_{p\geq n} x_p.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded below, the set of its cluster points (when it is not empty) is also bounded below, and the infimum of all the cluster points is finite and coincides with $\liminf x_n$. One then verifies that there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to $\liminf x_n$.

Similarly, if the sequence is bounded above and the set of cluster points is not empty, the supremum of the cluster points is finite and coincides with $\limsup x_n$.

Of course, the sequence converges if and only if $\liminf x_n$ and $\limsup x_n$ are finite and equal and, in this case, the limit of the sequence is equal to the common value of $\liminf x_n$ and $\limsup x_n$.

Finally, let us remind the reader that $\liminf x_n$ and $\limsup x_n$ keep their meaning in $\mathbb{R} \cup \{-\infty, +\infty\}$ for any real sequence. For example, if $(x_n)_{n\in\mathbb{N}}$ is not bounded above, we have $\limsup x_n = +\infty$.

§3.3 Continuity in Metric Spaces

In a metric space, the set of open (or closed) balls centered at a point constitute a neighborhood base of that point.

Consequently, a function f defined on a subset A of a metric space (X_1, d_1) and with values in a metric space (X_2, d_2) is continuous at $x \in A$ if and only if we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in X_1 \cap A, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon.$$

This will allow us to establish the following important property (which, also, is not true in "general" topological spaces).

Proposition 3.3.1

A map between two metric spaces is continuous at a point if and only if it is sequentially continuous at that point.

Proof. Only the converse needs to be shown, the direct implication being true in any topological space. Let us reason by contraposition. Let f be a function defined on a subset A of a metric space (X_1, d_1) and with values in a metric space (X_2, d_2). Let $a \in A$. Suppose that f is not continuous at a. Then there exists an $\varepsilon > 0$ such that for every neighborhood V of a there exists an $x \in V \cap A$ satisfying $d_2(f(x), f(a)) \geq \varepsilon$. Taking for V the open balls $B(a, 2^{-n})$, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, d_2(f(x_n), f(a)) \geq \varepsilon \quad \text{and} \quad d_1(x_n, a) < 2^{-n}.$$

Consequently, $(f(x_n))_{n\in\mathbb{N}}$ does not converge to $f(a)$ even though $(x_n)_{n\in\mathbb{N}}$ tends to a. ■

We have seen that for a map $f : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces, continuity at every point could be characterized as follows:

$$\forall \varepsilon > 0, \forall x \in X_1, \exists \delta > 0, \forall y \in X_1, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon.$$

We can define a stronger notion of continuity called uniform continuity:

Definition 3.3.1

We say that a map $f : X_1 \to X_2$ between two metric spaces (X_1, d_1) and (X_2, d_2) is **uniformly continuous** if we have:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X_1, \forall y \in X_1, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon.$$

Of course, uniform continuity implies continuity. The converse is false in general but true in the case of a compact metric space (see later).

Definition 3.3.2

Let k be a positive real number. A map f between two metric spaces (X_1, d_1) and (X_2, d_2) is said to be **Lipschitz** with constant k if

$$\forall (x, y) \in X_1 \times X_1,\ d_2(f(x), f(y)) \leq k d_1(x, y).$$

Example 3.5. *In $\mathbb{R}^+$ equipped with the distance associated with the absolute value, the function $x \mapsto 2x+3$ is Lipschitz with constant 2, the function $x \mapsto \sqrt{x}$ is uniformly continuous without being Lipschitz, and the function $x \mapsto x^2$ is continuous without being either uniformly continuous or Lipschitz.*

Exercise 3.4. *Show that Lipschitz implies uniformly continuous, and uniformly continuous implies continuous.*

Here is a particular example of a Lipschitz map.

Definition 3.3.3

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that $f : X_1 \to X_2$ is an **isometry** if f is bijective and preserves the distance:

$$\forall (x, y) \in X_1^2,\ d_2(f(x), f(y)) = d_1(x, y).$$

Remark 3.6. *The properties of continuity, uniform continuity, and being Lipschitz are invariant under changing the distance to an equivalent distance.*

§3.4 Compactness in Metric Spaces

In general, uniform continuity is a notion strictly stronger than continuity. The following theorem establishes that in a compact metric space, the two notions coincide.

Theorem 3.4.1: Heine's Theorem

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose (X, d_X) is compact. Then every continuous function from X to Y is uniformly continuous on X.

Proof. Fix $\varepsilon > 0$. By definition of continuity, for each $x \in X$, there exists $\eta_x > 0$ such that

$$d_X(x, y) < \eta_x \Longrightarrow d_Y(f(x), f(y)) < \frac{\varepsilon}{2}.$$

By compactness of X, we can find a finite number of points $x_1, \cdots, x_N$ such that

$$X \subset \bigcup_{k=1}^{N} B_X\left(x_k, \frac{\eta_{x_k}}{2}\right).$$

Set $\eta = \min_{1 \le k \le N} \eta_{x_k}$. For every pair (x, y) such that $d_X(x, y) < \frac{\eta}{2}$, we can find an index k such that x and y are in $B_X(x_k, \eta_{x_k})$. We then have

$$d_Y(f(x), f(y)) \le d_Y(f(x), f(x_k)) + d_Y(f(x_k), f(y)) < \varepsilon,$$

hence the uniform continuity. ∎

In metric spaces, compactness can be characterized using sequences. This fundamental fact is the subject of the theorem below.

Theorem 3.4.2: Bolzano-Weierstrass Theorem

A metric space (X, d) is compact if and only if every sequence of elements of X admits a convergent subsequence.

Proof. By Proposition 3 page 11, every sequence in a compact space admits a cluster point. Moreover, in a metric space, there is equivalence between having convergent subsequences and possessing cluster points, so the direct implication is established.

Let us now focus on the proof of the converse implication. Let X be a metric space in which every sequence has a cluster point. Let $(\Omega_i)_{i\in I}$ be an open cover of X. The goal is to extract from this family a finite subcover of X.

1. We reduce to a cover by open balls of the same radius. More precisely, we show that there exists an $\varepsilon > 0$ such that every open ball of radius ε is contained in one of the Ω_i. We proceed by contradiction by assuming that such an ε does not exist. Then for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that $B(x_n, 2^{-n})$ is contained in none of the Ω_i. This sequence admits a cluster point x which, inevitably, belongs to one of the Ω_i. Denote by i_x the corresponding index and $r > 0$ such that $B(x, r) \subset \Omega_{i_x}$. Since x is a cluster point of the sequence, there exists $n \in \mathbb{N}$ such that $d(x, x_n) < r/2$ and $2^{-n} \leq r/2$. We then have $B(x_n, 2^{-n}) \subset \Omega_{i_x}$, which contradicts the definition of x_n.

2. We show that X can be covered by a finite number of balls of the same radius ε. For this, start with an arbitrary point x_0. If $X \subset B(x_0, \varepsilon)$, there is nothing to do, the construction is finished. Otherwise, choose an x_1 such that $x_1 \notin B(x_0, \varepsilon)$. If $B(x_0, \varepsilon) \cup B(x_1, \varepsilon)$ covers X, the construction is finished, otherwise choose a point x_2 such that $x_2 \notin B(x_0, \varepsilon) \cup B(x_1, \varepsilon)$. By recurrence, we construct thus a family $(x_0, \cdots, x_n)$ such that $x_n \notin B(x_0, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)$. The construction process necessarily stops after a finite number of steps, otherwise we would obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$. Such a sequence could not have a cluster point.

3. Conclusion. By the first step, there exists $\varepsilon > 0$ such that for each $x \in X$ there exists an index $i_x \in I$ such that $B(x, \varepsilon) \subset \Omega_{i_x}$. By the second step, there exists a finite family $(x_1, \cdots, x_N)$ of elements of X such that $X \subset \bigcup_{k=1}^{N} B(x_k, \varepsilon)$. We then have $X \subset \bigcup_{k=1}^{N} \Omega_{i_{x_k}}$.

■

Let us give several important consequences of the Bolzano-Weierstrass theorem.

Definition 3.4.1

We say that a subset A of a Hausdorff topological space X is **relatively compact** if $\overline{A}$ is compact.

Corollary 3.4.1

Let (X, d) be a metric space. A subset A of X is relatively compact if and only if from every sequence of elements of A, one can extract a subsequence that converges in X.

Proof. Suppose A is relatively compact. Then every sequence of elements of A is also a sequence of elements of the compact set $\overline{A}$. Consequently, it admits at least one cluster point in $\overline{A}$, and hence a convergent subsequence in X. Conversely, suppose that from every sequence of A one can extract a convergent subsequence and consider a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\overline{A}$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of A such that $d(x_n, y_n) \leq 2^{-n}$ for all $n \in \mathbb{N}$. This sequence admits a convergent subsequence $(y_{\varphi(n)})_{n\in\mathbb{N}}$ and it is clear that $(x_{\varphi(n)})_{n\in\mathbb{N}}$ converges to the same limit. This limit is necessarily in the closed set $\overline{A}$. Consequently, A is indeed relatively compact. ■

Corollary 3.4.2

In a metric space, every relatively compact subset is bounded and every compact subset is closed and bounded.

Proof. To prove the first part of the corollary, let us reason by contraposition. Let A be an unbounded subset of the metric space (X, d). Then we can construct (by recurrence) a sequence $(x_n)_{n\in\mathbb{N}}$ such that $d(x_n, x_m) \geq 1$ for all $(n, m) \in \mathbb{N}^2$ with $n \neq m$. Such a sequence cannot have a convergent subsequence. The second part of the corollary then becomes evident. Indeed, every compact set is relatively compact (hence bounded), and closed. ■

Corollary 3.4.3

The Cartesian product of a finite number of compact metric spaces is a compact metric space.

Proof. To simplify, let us limit ourselves to the product of two compact metric spaces (X_1, d_1) and (X_2, d_2). Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence of elements of $X_1 \times X_2$ (equipped with the product metric). By compactness of X_1, the sequence $(x_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$. By compactness of X_2, the sequence $(y_{\varphi(n)})_{n\in\mathbb{N}}$ admits a convergent subsequence $(y_{\varphi(\psi(n))})_{n\in\mathbb{N}}$. It is clear that the sequence $(x_{\varphi(\psi(n))}, y_{\varphi(\psi(n))})_{n\in\mathbb{N}}$ is convergent. The converse of the Bolzano-Weierstrass theorem then allows us to conclude. ■

Remark 3.7. *Using Cantor's diagonal argument, one can establish that the product of a countable family of compact metric spaces is a compact metric space. This remains true for an arbitrary family if one uses the axiom of choice.*

§3.5 Completeness

Let us first recall the definition of a Cauchy sequence.

Definition 3.5.1

Let (X, d) be a metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of $X^{\mathbb{N}}$ is a **Cauchy sequence** if it satisfies:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \, (p \geq N \, \text{and} \, n \geq N) \implies d(x_n, x_p) < \varepsilon.$$

As an exercise, we leave it to the reader to establish that every convergent sequence is Cauchy. The converse is false in general. Indeed, consider the set $\mathbb{Q}$ equipped with the distance given by the absolute value. Then the sequence defined by

$$x_0 = 1 \, \text{and} \, x_{n+1} = \frac{1}{1 + x_n} \, \text{for} \, n \geq 1$$

is a Cauchy sequence in $\mathbb{Q}$ (because it converges in $\mathbb{R}$ for the same distance). However, its limit $\frac{1}{2}(1 + \sqrt{5})$ is irrational and hence this sequence does not converge in $\mathbb{Q}$. This example motivates the following definition:

Definition 3.5.2

The metric space (X, d) is said to be **complete** if every Cauchy sequence of elements of X converges in X. A subset A of a metric space is said to be **complete** if, equipped with the induced distance, it is a complete metric space.

Example 3.6. *The set of real numbers equipped with the usual distance is, by construction, complete. On the other hand, $\mathbb{Q}$ equipped with the same distance is not complete.*

Exercise 3.5. *Verify that in any metric space, a Cauchy sequence having a cluster point is convergent.*

Proposition 3.5.1

Let (X, d) and (X', d') be two metric spaces. Suppose there exists a bijective map $f : X \to X'$ such that f and f^{-1} are uniformly continuous. Then (X, d) is complete if and only if (X', d') is complete.

Proof. Suppose (X, d) is complete. Let $(x'_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X', d'). Since f^{-1} is uniformly continuous, one verifies (by going back to the definition) that $(f^{-1}(x'_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). But since the space (X, d) is complete, this sequence converges to an element x of X. Finally, since f is continuous, we conclude that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

The converse is treated by exchanging the roles of X (resp. f) and X' (resp. f^{-1}). ■

Remark 3.8. *Contrary to the topological properties seen so far, completeness is not preserved by homeomorphism.*

Proposition 3.5.2

Every complete subset is closed.

Proof. Let A be a complete subset of the metric space (X, d). If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of A then it is a Cauchy sequence. Since A is complete, the limit of this sequence is in A. Consequently A is closed. ■

Proposition 3.5.3

Every closed subset of a complete metric space is complete.

Proof. Let A be a closed subset of a complete metric space (X, d) and $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in A. Then $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X and hence converges in X. Since A is closed, the limit of this sequence is in A. ■

We leave it to the reader to show the following result:

Proposition 3.5.4

The Cartesian product of a finite number of complete metric spaces is a complete metric space.

The following proposition allows us to compare the notions of completeness and compactness.

Proposition 3.5.5

Let (X, d) be a metric space. The following two properties are equivalent:

1. (X, d) is compact,

2. (X, d) is complete and, for every $\varepsilon > 0$, X can be covered by a finite number of balls of radius ε.

Proof. (i) $\Rightarrow$ (ii): Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Since (X, d) is a compact metric space, the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point. But since it is Cauchy, it converges. Consequently (X, d) is complete. Moreover, compactness also ensures that X can be covered by a finite number of open balls of any given radius.

(ii) $\Rightarrow$ (i): Suppose that (X, d) is complete and that, for every $\varepsilon > 0$, X can be covered by a finite number of open balls of radius ε.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X^{\mathbb{N}}$. The goal is to show that this sequence admits a convergent subsequence. In fact, we will rather establish that there exists a subsequence $(x_{\psi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, and a sequence $(y_n)_{n\in\mathbb{N}}$ in $X^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ and $p \in \mathbb{N}$ we have $x_{\psi(n+p)} \in B(y_n, 2^{-n})$. This will ensure that $(x_{\psi(n)})_{n\in\mathbb{N}}$ is Cauchy and hence converges since (X, d) is complete.

In order to construct the $(y_n)_{n\in\mathbb{N}}$ and the successive extractions, we start by remarking that since X is covered by a *finite* number of balls of radius 1, there exists $y_0 \in X$ such that $B(y_0, 1)$ contains infinitely many terms of $(x_n)_{n\in\mathbb{N}}$. This allows us to define a subsequence $(x_{\varphi_0(n)})_{n\in\mathbb{N}} \in B(y_0, 1)^{\mathbb{N}}$. Similarly, X can be covered by a finite number of balls of radius $1/2$ so there exists $y_1 \in X$ such that the ball $B(y_1, 1/2)$ contains infinitely many terms of $(x_{\varphi_0(n)})_{n\in\mathbb{N}}$. This ensures the existence of a second extraction φ_1 such that $B(y_1, 1/2)$ contains all the terms of the sequence $(x_{\varphi_0(\varphi_1(n))})_{n\in\mathbb{N}}$. By an elementary recurrence, we thus obtain a sequence $(y_n)_{n\in\mathbb{N}}$ of points of X and extractions $\varphi_0, \cdots, \varphi_k, \cdots$ such that $B(y_k, 2^{-k})$ contains all the terms of $(x_{\varphi_0\circ\cdots\circ\varphi_k(n)})_{n\in\mathbb{N}}$. We conclude using the *diagonal process* which consists in setting $\psi(n) = \varphi_0 \circ \cdots \circ \varphi_n(n)$. The subsequence thus constructed satisfies the desired property. ■

Corollary 3.5.1

Let (X, d) be a complete metric space and A a subset of X. The following two statements are equivalent:

1. the subset A is relatively compact,

2. for every $\varepsilon > 0$, the subset A can be covered by a finite number of balls centered at points of A.

Proof. (i) $\Rightarrow$ (ii): It suffices to note that

$$\overline{A} \subset \bigcup_{x\in A} B(x, \varepsilon).$$

Since $\overline{A}$ is compact, we can extract from the above cover a finite subcover of $\overline{A}$ and hence (a fortiori) of A.

(ii) $\Rightarrow$ (i): We note that the completeness of (X, d) ensures that $(\overline{A}, d)$ is complete. It is clear that if for every $\varepsilon > 0$, the subset A can be covered by a finite number of balls centered at points of A, the same holds for $\overline{A}$. Indeed

$$A \subset \bigcup_{i=1}^{n} B\Big(x_i, \frac{\varepsilon}{2}\Big) \Longrightarrow \overline{A} \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

The previous proposition allows us to conclude that $(\overline{A}, d)$ is compact. ■

Corollary 3.5.2

The compact subsets of $\mathbb{R}$ are the closed bounded sets.

Proof. We already know that in a metric space, compact sets are closed and bounded. Conversely, knowing that $\mathbb{R}$ is complete and that every bounded subset of $\mathbb{R}$ can be covered by a finite number of balls of arbitrarily fixed radius ε, the previous corollary ensures that every bounded subset of $\mathbb{R}$ is relatively compact. ■

Remark 3.9. *There exist complete metric spaces that are not compact:* $\mathbb{R}$ *for example.*

In a complete space, we have the following extension result:

Theorem 3.5.1

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose (Y, d_Y) is complete. Let A be a dense subset of X and $f : A \to Y$ uniformly continuous.

Then there exists a unique map $\bar{f} : X \to Y$ continuous on X that extends f on X. Moreover, this extension is uniformly continuous on X.

Proof. Let us first prove uniqueness. Let f_1 and f_2 be two continuous extensions of f, and $x \in X$ arbitrary. Fix $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ converging to x. Since each x_n is in A, we have $f_1(x_n) = f_2(x_n)$. Passing to the limit, we conclude that $f_1(x) = f_2(x)$.

Now let us pass to the existence of the extension. Let $x \in X$ be arbitrary and $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ converging to x. The sequence $(x_n)_{n\in\mathbb{N}}$ is thus Cauchy in (X, d_X). Using the uniform continuity of f, one easily verifies that $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in the complete metric space (Y, d_Y) and hence converges to some limit ℓ. We leave it to the reader to verify that the value of ℓ does not depend on the choice of the sequence $(x_n)_{n\in\mathbb{N}}$ converging to x. We then set $\bar{f}(x) = \ell$.

By passing to the limit in the definition of the uniform continuity of f, we conclude the uniform continuity of $\bar{f}$ (in fact, for a fixed ϵ, any η that works for f also works for $\bar{f}$). ■

Definition 3.5.3

Let (X, d) be a metric space. We say that a map f from X to X is **contractive** if there exists $k \in [0, 1[$ such that

$$\forall (x, y) \in X^2,\ d(f(x), f(y)) \leq kd(x, y).$$

Theorem 3.5.2: Fixed point theorem

Let (X, d) be a complete metric space and f a contractive map from X to X. Then f admits a unique fixed point.

Proof. The proof is at least as interesting as the statement of the theorem. Fix a real number $k \in [0, 1[$ such that

$$d(f(x), f(y)) \leq kd(x, y) \quad \text{for all} \quad (x, y) \in X^2. \tag{2.2}$$

The uniqueness of the fixed point is evident. Indeed if $f(x) = x$ and $f(y) = y$ then (2.2) ensures that $d(x, y) \leq kd(x, y)$, whence $(1 - k)d(x, y) \leq 0$ and then $d(x, y) = 0$.

To show the existence of the fixed point, start with an arbitrary $x_0 \in X$ and define the sequence $(x_n)_{n \in \mathbb{N}}$ by the recurrence relation $x_{n+1} = f(x_n)$. Exploiting (2.2), we see that

$$\forall n \in \mathbb{N},\ d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$$

and then that

$$\forall (n, p) \in \mathbb{N}^2,\ d(x_{n+p}, x_n) \leq \frac{k^n}{1 - k} d(x_1, x_0).$$

Consequently, the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy. Since (X, d) is complete, it converges to a point x of X which, clearly, satisfies $f(x) = x$. ■

Chapter 4

Normed Vector Spaces

§4.1 Definitions

Definition 4.1.1

Let E be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : E \to \mathbb{R}^+$ is called a **norm** on E if it satisfies the following conditions:

1. $\|x\| = 0 \iff x = 0$,
2. $\forall \lambda \in \mathbb{K}, \forall x \in E, \|\lambda x\| = |\lambda| \|x\|$,
3. $\forall (x, y) \in E^2, \|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is then called a **normed vector space**.

Remark 4.1. *On $E \times E$, we define the function d by:*

$$\forall (x, y) \in E^2, \, d(x, y) = \|x - y\|.$$

*One easily verifies that d is a distance on E. It is called the **distance associated to the norm**. In addition to the three properties defining distances, the function d is translation invariant:*

$$\forall (x, y, z) \in X^3, \, d(x + z, y + z) = d(x, y)$$

and homogeneous of degree one:

$$\forall (x, y, \lambda) \in X \times X \times \mathbb{K}, \, d(\lambda x, \lambda y) = |\lambda| d(x, y).$$

*The **reverse triangle inequality** for d can be written in terms of the norm:*

$$\forall (x, y) \in E^2, \, |\|x\| - \|y\|| \leq \|x - y\|.$$

In all that follows, we equip the normed vector space E with the topology associated to the distance d. Note that balls and spheres can be defined in terms of the norm:

- *the **open ball** $B_E(x_0, r)$ with center x_0 and radius r is equal to $\{x \in E \mid \|x - x_0\| < r\}$,*
- *the **closed ball** $\overline{B}_E(x_0, r)$ with center x_0 and radius r is equal to $\{x \in E \mid \|x - x_0\| \leq r\}$,*

- *the **sphere** $S_E(x_0, r)$ with center x_0 and radius r is equal to* $\{x \in E \mid \|x - x_0\| = r\}$.

Exercise 4.1. *Let E be a normed vector space, $r > 0$ and $x_0 \in E$. Show that the closure of $B_E(x_0, r)$ is equal to $\overline{B}_E(x_0, r)$ and that the interior of $\overline{B}_E(x_0, r)$ is $B_E(x_0, r)$. Deduce that the boundary of $B_E(x_0, r)$ is the sphere $S_E(x_0, r)$. What remains of these results in an "arbitrary" metric space?*

Example 4.1. *1. Let $E = \mathbb{R}^n$. It is easy to establish that the function $x \mapsto \|x\|_\infty = \sup_{i=1}^n |x_i|$ is a norm on $\mathbb{R}^n$.*

Now fix a real number $p \geq 1$. For $x \in \mathbb{R}^n$, set

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

Then $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. The triangle inequality

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

*which is not trivial if $p > 1$ is called **Minkowski's inequality**.*

Minkowski's inequality is shown using Hölder's inequality valid on $\mathbb{R}^n$ or $\mathbb{C}^n$:

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \|y\|_q$$

where $1 \leq p \leq +\infty$ and q is the conjugate exponent of p defined by the relation $1/p + 1/q = 1$ (with the convention $1/\infty = 0$).

The case $p = q = 2$ is none other than the Cauchy-Schwarz inequality. The general case relies on Young's inequality:

$$\forall (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+,\ ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Indeed, if we exclude the trivial cases where x or y is zero, we can divide by $\|x\|_p \|y\|_q$ and we have, by Young's inequality:

$$\forall i \in \{1, \cdots, n\},\ \frac{|x_i|}{\|x\|_p} \frac{|y_i|}{\|y\|_q} \leq \frac{1}{p} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q}.$$

A simple summation over i gives Hölder's inequality.

Let us return to the proof of Minkowski's inequality. Applying Hölder's inequality twice, we obtain:

$$\begin{aligned}
\|x + y\|_p^p &= \sum_{i=1}^n |x_i + y_i| |x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\
&\leq \|x\|_p \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} + \|y\|_p \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{p-1}{p}} \\
&= (\|x\|_p + \|y\|_p) \|x + y\|_p^{p-1}.
\end{aligned}$$

Example 4.2. *For every $p \in [1, +\infty]$, we can equip the set $\ell^p(\mathbb{K})$ of sequences in $\mathbb{K}$ with p-th power summable with a normed vector space structure by setting:*

$$\|x\|_{\ell^p} = \left(\sum_{n=0}^{+\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

The triangle inequality is demonstrated as in the previous example by letting n tend to infinity.

The set $\ell^\infty(\mathbb{K})$ of bounded sequences in $\mathbb{K}$ can be equipped with the norm $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n|$.

Example 4.3. *The set $C([0,1];\mathbb{R})$ can be equipped with a norm by setting*

$$\|f\|_{L^\infty} \stackrel{def}{=} \sup_{t\in[0,1]} |f(t)|.$$

*The norm $\|\cdot\|_{L^\infty}$ thus defined is called the **uniform norm**. Other choices are possible. For example*

$$\|f\|_{L^2} \stackrel{def}{=} \sqrt{\int_0^1 |f(t)|^2 dt},$$

$$\|f\|_{L^1} \stackrel{def}{=} \int_0^1 |f(t)| dt,$$

$$\|f\|_{L^p} \stackrel{def}{=} \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}$$

with $p \in [1, +\infty[$.

Example 4.4. *The set $\mathcal{M}_n(\mathbb{R})$ of square matrices of size n with real coefficients can be equipped with the norms*

$$\|A\|_\infty = \max_{1\le i,j\le n} |a_{ij}| \quad \textit{or} \quad \|A\|_p = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^p \right)^{\frac{1}{p}} \quad \textit{with} \quad p \in [1, +\infty[.$$

Definition 4.1.2

Let E be a vector space equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. We say that $\|\cdot\|_2$ is **stronger** than $\|\cdot\|_1$ if there exists a constant $c > 0$ such that

$$\forall x \in E, \quad \|x\|_1 \le c\|x\|_2.$$

We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if there exists $c > 0$ such that

$$\forall x \in E, \quad c^{-1}\|x\|_2 \le \|x\|_1 \le c\|x\|_2.$$

It is clear that the distances associated to two equivalent norms are equivalent. Consequently, we have:

Proposition 4.1.1

The topologies generated by two equivalent norms are the same.

As an exercise, the reader may verify the following proposition:

Proposition 4.1.2

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. Then $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ if and only if every open set of E for the topology associated to $\|\cdot\|_1$ is open for the topology of $\|\cdot\|_2$.

We will remember that the stronger the norm, the finer the associated topology (that is, the more open sets it has).

Example 4.5. *1. In $\mathcal{M}_n(\mathbb{R})$, the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent.*

2. In $C([0,1];\mathbb{R})$, the norm $\|\cdot\|_{L^\infty}$ is (strictly) stronger than $\|\cdot\|_{L^1}$.

Remark 4.2. *If $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are two normed vector spaces, we can equip $E \times F$ with a* ***product norm*** *by setting for every $(u,v) \in E \times F$,*

$$\|(u,v)\|_1 = \|u\|_E + \|v\|_F,$$
$$\|(u,v)\|_2 = \sqrt{\|u\|_E^2 + \|v\|_F^2},$$
$$\|(u,v)\|_\infty = \max(\|u\|_E, \|v\|_F).$$

The three norms above are equivalent. The associated topologies are therefore the same. We leave it to the reader to define a norm for the product of a finite number of normed vector spaces.

§4.2 Linear Maps

Definition 4.2.1

Let E and F be two normed vector spaces over the same field $\mathbb{K}$. We denote by $L(E;F)$ the set of linear maps from E to F.

The equivalence between the first two assertions of the following proposition will be fundamental for the rest of the course.

Proposition 4.2.1

Let $u \in L(E; F)$. The following five properties are equivalent.

1. $\exists M \geq 0, \forall x \in E, \|u(x)\|_F \leq M\|x\|_E$,
2. u is continuous on E,
3. u is continuous at 0,
4. u is bounded on the closed unit ball,
5. u is bounded on the unit sphere.

Proof. It suffices to prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ By linearity of u, we have $u(y) - u(x) = u(y - x)$. Therefore,

$$\forall (x, y) \in E^2, \ \|u(y) - u(x)\|_F \leq M\|y - x\|_E.$$

So u is Lipschitz and hence continuous.

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (iv)$ From the continuity of u at 0, we deduce the existence of an $\eta > 0$ such that $\|u(x)\|_F \leq 1$ whenever $x \in B_E(0, \eta)$. Now $x \in B_E(0, \eta)$ if and only if $\eta^{-1}x \in B_E(0, 1)$. By linearity of u, we conclude that $\|u(x)\|_F \leq \eta^{-1}$ for all $x \in B_E(0, 1)$. So u is bounded on the unit ball.

$(iv) \Rightarrow (v)$ Trivial.

$(v) \Rightarrow (i)$ Let M be a bound of u restricted to the unit sphere. If $x \neq 0$, the point $x/\|x\|_E$ belongs to the unit sphere. Using the linearity of u and the fact that u is bounded by M on the unit sphere, we thus obtain

$$\|u(x)\|_F = \|x\|_E \left\| u\left(\frac{x}{\|x\|_E}\right) \right\|_F \leq M\|x\|_E.$$

■

Exercise 4.2. *Using the above proposition, show that the map*

$$\begin{cases} E \times E & \longrightarrow E \\ (x, y) & \longmapsto x + y \end{cases}$$

is continuous.

Definition 4.2.2

Let $\mathcal{L}(E, F)$ be the set of continuous linear maps from E to F. For $f \in \mathcal{L}(E, F)$, we denote by $\|f\|_{\mathcal{L}(E,F)}$ the smallest constant M such that

$$\forall x \in E, \ \|f(x)\|_F \leq M\|x\|_E.$$

We thus have

$$\forall x \in E, \ \|f(x)\|_F \leq \|f\|_{\mathcal{L}(E,F)}\|x\|_E.$$

The reader will easily verify that we also have

$$\|f\|_{\mathcal{L}(E,F)} = \sup_{x\in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E} = \sup_{\substack{x\in E\\ \|x\|_E=1}} \|f(x)\|_F.$$

Proposition 4.2.2

The space $(\mathcal{L}(E,F); \|\cdot\|_{\mathcal{L}(E,F)})$ is a normed vector space.

Proof. Let us quickly verify that $\|\cdot\|_{\mathcal{L}(E,F)}$ is a norm. Proposition 5 ensures that for every $f \in \mathcal{L}(E,F)$, the quantity $\|f\|_{\mathcal{L}(E,F)}$ is finite (and positive). It is moreover immediate that $\|\lambda f\|_{\mathcal{L}(E,F)} = |\lambda|\|f\|_{\mathcal{L}(E,F)}$ for every $\lambda \in \mathbb{K}$ and that $\|f\|_{\mathcal{L}(E,F)} = 0$ if and only if $f = 0$.

If f and g are two elements of $\mathcal{L}(E,F)$, we have for every $x \in E$,

$$\begin{aligned}\|(f+g)(x)\|_F &= \|f(x)+g(x)\|_F \\ &\leq \|f(x)\|_F + \|g(x)\|_F \\ &\leq (\|f\|_{\mathcal{L}(E,F)} + \|g\|_{\mathcal{L}(E,F)})\|x\|_E.\end{aligned}$$

So $f+g$ is linear continuous and satisfies $\|f+g\|_{\mathcal{L}(E,F)} \leq \|f\|_{\mathcal{L}(E,F)} + \|g\|_{\mathcal{L}(E,F)}$. So $\|\cdot\|_{\mathcal{L}(E,F)}$ is indeed a norm. ■

Remark 4.3. *The definition of $\mathcal{L}(E;F)$ depends on the choice of norms on E and F.*

To illustrate this fact, take $E = \mathcal{C}([0,1];\mathbb{R})$ and $F = \mathbb{R}$. We equip F with the norm given by the absolute value. Consider the linear form L defined on E by $L(f) = f(0)$.

It is immediate that L is continuous if we equip E with the norm $\|\cdot\|_{L^\infty}$. On the other hand, L is not continuous if we equip E with the norm $\|\cdot\|_{L^1}$. To be convinced of this, one may consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n(x) = \begin{cases} n - n^2 x & \text{if } x \in [0, \frac{1}{n}], \\ 0 & \text{if } x \in [\frac{1}{n}, 1]. \end{cases}$$

However, changing the norm on E and the norm on F to equivalent norms does not modify $\mathcal{L}(E;F)$, and changes $\|\cdot\|_{\mathcal{L}(E,F)}$ to an equivalent norm.

Proposition 4.2.3

Let E, F and G be three normed vector spaces, $f \in \mathcal{L}(E;F)$ and $g \in \mathcal{L}(F;G)$. Then $g \circ f \in \mathcal{L}(E;G)$ and we have the following inequality:

$$\|g \circ f\|_{\mathcal{L}(E;G)} \leq \|f\|_{\mathcal{L}(E;F)}\|g\|_{\mathcal{L}(F;G)}.$$

Proof. First of all, the theorem of composition of continuous maps ensures that $g \circ f$ is continuous. The linearity of $g \circ f$ is evident. Let $x \in E$. By definition of $\|g\|_{\mathcal{L}(F;G)}$, we have

$$\|g \circ f(x)\|_G \leq \|g\|_{\mathcal{L}(F;G)} \|f(x)\|_F,$$

and then by definition of $\|f\|_{\mathcal{L}(E;F)}$,

$$\|g \circ f(x)\|_G \leq \|g\|_{\mathcal{L}(F;G)} \|f\|_{\mathcal{L}(E;F)} \|x\|_E.$$

Consequently, we indeed have $\|g \circ f\|_{\mathcal{L}(E;G)} \leq \|f\|_{\mathcal{L}(E;F)} \|g\|_{\mathcal{L}(F;G)}$. ∎

Remark 4.4. *The above proposition ensures that the set $\mathcal{L}(E)$ of continuous linear maps from E to E is such that:*

$$\forall (f, g) \in \mathcal{L}(E), \|g \circ f\|_{\mathcal{L}(E)} \leq \|f\|_{\mathcal{L}(E)} \|g\|_{\mathcal{L}(E)}.$$

One easily verifies that $\|\mathrm{Id}_E\|_{\mathcal{L}(E)} = 1$.

These two additional properties confer to the normed vector space $(\mathcal{L}(E); \|\cdot\|_{\mathcal{L}(E)})$ a structure of a normed algebra. We say that $\|\cdot\|_{\mathcal{L}(E)}$ is an algebra norm.

For multilinear maps, we can establish a characterization of continuity similar to the one we have for linear maps:

Proposition 4.2.4

Let $E_1, \cdots, E_k$ and F be vector spaces over $\mathbb{K}$, equipped with norms $\|\cdot\|_{E_1}, \cdots, \|\cdot\|_{E_k}$ and $\|\cdot\|_F$. Let u be a k-linear map from $E_1 \times \cdots \times E_k$ to F.

The following four properties are equivalent:

1. u is continuous,
2. u is continuous at $(0, \cdots, 0)$,
3. u is bounded on $\overline{B}_{E_1}(0,1) \times \cdots \times \overline{B}_{E_k}(0,1)$,
4. there exists $M \in \mathbb{R}^+$ such that

$$\begin{aligned} &\forall (x_1, \cdots, x_k) \in E_1 \times \cdots \times E_k, \\ &\|u(x_1, \cdots, x_k)\|_F \leq M \|x_1\|_{E_1} \cdots \|x_k\|_{E_k}. \end{aligned} \tag{3.2}$$

Proof. To simplify the presentation, we assume that $k = 2$ and we equip $E_1 \times E_2$ with the product norm

$$\|(x_1, x_2)\|_{E_1 \times E_2} = \max(\|x_1\|_{E_1}, \|x_2\|_{E_2}).$$

It suffices to prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$. $(i) \Rightarrow (ii)$ Obvious. $(ii) \Rightarrow (iii)$ From the continuity at $(0,0)$, we deduce the existence of an $\eta > 0$ such that $\|x_1\|_{E_1} \leq \eta$ and $\|x_2\|_{E_2} \leq \eta$ implies $\|u(x_1, x_2)\|_F \leq 1$. We deduce that

$$\|(x_1, x_2)\|_{E_1 \times E_2} \leq 1 \implies \|u(x_1, x_2)\|_F \leq \eta^{-2}.$$

(iii) $\Rightarrow$ *(iv)* Let M be a bound of u on $\overline{B}_{E_1}(0,1) \times \overline{B}_{E_2}(0,1)$. Let $(x_1, x_2) \in E_1 \times E_2$ such that $x_1 \neq 0$ and $x_2 \neq 0$. We then have $\frac{x_1}{\|x_1\|_{E_1}}, \frac{x_2}{\|x_2\|_{E_2}} \in \overline{B}_{E_1}(0,1) \times \overline{B}_{E_2}(0,1)$ so

$$\|u(x_1,x_2)\|_F = \|x_1\|_{E_1}\|x_2\|_{E_2}\left\|u\left(\frac{x_1}{\|x_1\|_{E_1}}, \frac{x_2}{\|x_2\|_{E_2}}\right)\right\|_F \leq M\|x_1\|_{E_1}\|x_2\|_{E_2}.$$

(iv) $\Rightarrow$ *(i)* Let (x_1, x_2) and (y_1, y_2) be two elements of $E_1 \times E_2$. By bilinearity of u, we have

$$u(y_1,y_2) - u(x_1,x_2) = u(y_1 - x_1, y_2) + u(x_1, y_2 - x_2),$$

so

$$\|u(y_1,y_2) - u(x_1,x_2)\|_F \leq M\left(\|y_1 - x_1\|_{E_1}\|y_2\|_{E_2} + \|x_1\|_{E_1}\|y_2 - x_2\|_{E_2}\right).$$

This visibly ensures continuity at (x_1, x_2). ■

Exercise 4.3. *Using the above proposition, show that the map* $\begin{cases} \mathbb{K} \times E & \longrightarrow E \\ (\lambda, x) & \longmapsto \lambda x \end{cases}$ *is continuous.*

Definition 4.2.3

We denote by $\mathcal{L}^k(E_1 \times \cdots \times E_k; F)$ the set of continuous k-linear maps from $E_1 \times \cdots \times E_k$ to F.

For $u \in \mathcal{L}^k(E_1 \times \cdots \times E_k; F)$, we denote by $\|\cdot\|_{\mathcal{L}^k(E_1 \times \cdots \times E_k; F)}$ the infimum of all constants M satisfying (3.2). One can verify that this is a norm on $\mathcal{L}^k(E_1 \times \cdots \times E_k; F)$.

§4.3 Vector Subspaces

Proposition 4.3.1

Let $(E, \|\cdot\|_E)$ be a normed vector space and F a vector subspace of E. Then the closure of F in E is a vector subspace of E.

Proof. This is immediate using the characterization of closure by sequences. ■

Here is a very useful result on the extension of continuous linear maps:

Theorem 4.3.1

Let $(E, \|\cdot\|_E)$ be a normed vector space, $(G, \|\cdot\|_G)$ a complete normed vector space and F a dense vector subspace of E. Let $L \in \mathcal{L}(F;G)$. There exists a unique map $\tilde{L} \in \mathcal{L}(E;G)$ that extends L on E. Moreover, we have

$$\|\tilde{L}\|_{\mathcal{L}(E;G)} = \|L\|_{\mathcal{L}(F;G)}.$$

Proof. Knowing that every continuous linear map is in fact Lipschitz and hence uniformly continuous, the theorem on the extension of uniformly continuous maps ensures the existence and uniqueness of a continuous extension $\tilde{L}$ defined on E. Let us verify the linearity of $\tilde{L}$. So let $(\lambda, \mu) \in \mathbb{K}^2$, $(x, y) \in E^2$, $(x_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$ a sequence tending to x and $(y_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$ a sequence tending to y. Passing to the limit in the equality $L(\lambda x_n + \mu y_n) = \lambda L(x_n) + \mu L(y_n)$, we deduce that $\tilde{L}(\lambda x + \mu y) = \lambda \tilde{L}(x) + \mu \tilde{L}(y)$. Finally, knowing that

$$\|\tilde{L}\|_{\mathcal{L}(E;G)} = \sup_{x\in E\setminus\{0\}} \frac{\|\tilde{L}(x)\|_G}{\|x\|_E} \quad \text{and} \quad \|L\|_{\mathcal{L}(F;G)} = \sup_{x\in F\setminus\{0\}} \frac{\|L(x)\|_G}{\|x\|_E},$$

and that $L(x) = \tilde{L}(x)$ for $x \in F$, it is clear that $\|\tilde{L}\|_{\mathcal{L}(E;G)} \geq \|L\|_{\mathcal{L}(F;G)}$. But if $x \in E$ and $(x_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$ tends to x then we have

$$\forall n \in \mathbb{N}, \|\tilde{L}(x_n)\|_G \leq \|L\|_{\mathcal{L}(F;G)} \|x_n\|_E,$$

so

$$\|\tilde{L}(x)\|_G \leq \|L\|_{\mathcal{L}(F;G)} \|x\|_E.$$

This ensures that $\|\tilde{L}\|_{\mathcal{L}(E;G)} \leq \|L\|_{\mathcal{L}(F;G)}$. ■

Theorem 4.3.2

Let f be a linear form on $(E, \|\cdot\|_E)$. Then f is continuous if and only if $\ker f$ is closed.

Proof. If f is continuous then the set $\ker f$ is closed as the preimage of the closed set $\{0\}$ of $\mathbb{K}$. This establishes the direct implication. To establish the converse, suppose that f is not continuous. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ tending to 0 and such that $f(x_n) = 1$ for all $n \in \mathbb{N}$. Indeed, by Proposition 5, the linear form f is not continuous at 0. So there exists $\varepsilon > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$ tending to 0 such that $|f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = y_n/f(y_n)$ satisfies the desired properties. Now let $z \in E$ not belonging to $\ker f$ (this is possible because f is not zero since it is not continuous). We observe that the sequence with general term $z - f(z)x_n$ is in $(\ker f)^{\mathbb{N}}$ and converges to z. ■

From the above proof actually follows a much more precise result, namely:

Corollary 4.3.1

A nonzero linear form on a normed vector space E is not continuous if and only if its kernel is dense in E.

Theorem 4.3.3: Riesz's theorem

Let $(E, \|\cdot\|_E)$ be a normed vector space and M a closed vector subspace of E distinct from E. Then for every $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that

$$\|x_\varepsilon\|_E = 1 \quad \text{and} \quad \inf_{m\in M} \|x_\varepsilon - m\| \geq 1 - \varepsilon.$$

Proof. Let $y \in E \setminus M$. Set $\alpha \stackrel{\text{def}}{=} d(y, M)$. Since M is closed and $y \notin M$, we have $\alpha > 0$. For $\varepsilon \in [0, 1[$ fixed, we then choose $m_\varepsilon \in M$ such that $\|y - m_\varepsilon\|_E \leq \alpha/(1-\varepsilon)$. An easy calculation shows that the point $x_\varepsilon \stackrel{\text{def}}{=} \frac{y-m_\varepsilon}{\|y-m_\varepsilon\|_E}$ answers the question. ■

Remark 4.5. *In a Euclidean space, by using orthogonality, one can easily construct an $x \in E$ corresponding to $\varepsilon = 0$. But for a "general" normed vector space, one cannot do better than Riesz's theorem.*

§4.4 Banach Spaces

Definition 4.4.1

A complete normed vector space is called a **Banach space**.

Example 4.6.

1. *We will see in Chapter 5 that the normed vector space $C([0,1];\mathbb{R})$ equipped with the norm $\|\cdot\|_{L^\infty}$ is complete.*

2. *On the other hand, the space $C([0,1];\mathbb{R})$ equipped with the norm $\|\cdot\|_{L^1}$ is not complete. To see this, one can for example consider the sequence $(f_n)_{n\geq 2}$ defined by*

$$f_n(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2], \\ n(\frac{1}{2} + \frac{1}{n} - t) & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}], \\ 0 & \text{if } t \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

   It is Cauchy with respect to the norm $\|\cdot\|_{L^1}$ but does not converge in $C([0,1];\mathbb{R})$.

3. *We will see in the next section that all finite-dimensional normed vector spaces are complete.*

Recall that a series $\sum u_n$ of elements of E is said to be **absolutely convergent** if $\sum \|u_k\|$ converges.

Proposition 4.4.1

In a Banach space, every absolutely convergent series is convergent.

Proof. By virtue of the completeness property, it suffices to establish that the sequence with general term $\sum_{k=0}^{n} u_k$ is Cauchy. This follows from the fact that for all $m > n$, we have

$$\left\| \sum_{k=0}^{m} u_k - \sum_{k=0}^{n} u_k \right\| \leq \sum_{k=n+1}^{m} \|u_k\|,$$

and that $\sum \|u_k\|$ converges. ■

Exercise 4.4. *Show that, conversely, if E is a normed vector space in which every absolutely convergent series is convergent, then E is complete.*

Let us give a very useful completeness result:

Proposition 4.4.2

Let $(E, \|\cdot\|_E)$ be a normed vector space and $(F, \|\cdot\|_F)$ a Banach space. Then $(\mathcal{L}(E;F); \|\cdot\|_{\mathcal{L}(E;F)})$ is a Banach space.

Proof. We only give the structure of the proof and leave the pleasure of writing the details to the reader. So let $(L_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{L}(E;F)$. The goal is to show convergence to an element L of $\mathcal{L}(E;F)$.

1. **Pointwise convergence:** We show that for every $x \in E$ the sequence $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence of elements of F. Since F is complete, it converges to an element $L(x)$ of F.

2. **Study of the limit:** We first verify that $L \in L(E;F)$ and then that L is continuous.

3. **Convergence in norm:** We verify that $(L_n)_{n\in\mathbb{N}}$ converges to L in $\mathcal{L}(E;F)$.

■

Corollary 4.4.1

Let E be a normed vector space (arbitrary). The set of continuous linear forms on E is a Banach space.

Definition 4.4.2

The set of continuous linear forms on E is called the **topological dual** of E, and denoted E'. The norm $\|f\|'_E$ of an element of E' is thus defined by

$$\|f\|'_E = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_E}.$$

§4.5 The Case of Finite Dimension

The following result is fundamental:

Theorem 4.5.1

In a finite-dimensional vector space, the compact sets are the closed bounded sets and all norms are equivalent.

Proof. We have already seen that a compact set is closed and bounded. It therefore suffices to establish that in finite dimension every closed bounded set is compact and that all norms are equivalent. So let E be a finite-dimensional vector space of dimension p and $(e_1, \cdots, e_p)$ a basis of E. In all that follows, we denote

$$\|x\|_E^\infty = \max_{1 \leq i \leq p} |x_i| \quad \text{for} \quad x = \sum_{i=1}^{p} x_i e_i.$$

First step: We show that in $(E, \|\cdot\|_E^\infty)$ the closed bounded sets are compact. So let A be a closed bounded subset of E in the sense of the norm $\|\cdot\|_E^\infty$. Assume for simplicity that $\mathbb{K} = \mathbb{R}$ (the adaptation to the case $\mathbb{K} = \mathbb{C}$ is left to the reader). Let

$$\varphi : \begin{cases} (\mathbb{R}^p, \|\cdot\|_\infty) \longrightarrow (E, \|\cdot\|_E^\infty) \\ (x_1, \cdots, x_p) \longmapsto \sum_{i=1}^{p} x_i e_i. \end{cases}$$

Then φ is an isometric isomorphism between $(\mathbb{R}^p, \|\cdot\|_\infty)$ and $(E, \|\cdot\|_E^\infty)$. In particular, φ^{-1} is continuous, so $\varphi^{-1}(A)$ is a closed bounded subset of $(\mathbb{R}^p, \|\cdot\|_\infty)$. Let $M \geq 0$ such that $\varphi^{-1}(A) \subset [-M, M]^p$. The set $[-M, M]^p$ is compact as a Cartesian product of compacts. Since $\varphi^{-1}(A)$ is closed, we conclude that $\varphi^{-1}(A)$ is compact, and then that $A = \varphi(\varphi^{-1}(A))$ is also compact.

Second step: Equivalence of norms. Let now $\|\cdot\|_E$ be an arbitrary norm on E. Define:

$$\Psi : \begin{cases} (E, \|\cdot\|_E^\infty) \longrightarrow (\mathbb{R}, |\cdot|) \\ x \longmapsto \|x\|_E. \end{cases}$$

For $(x, y) \in E^2$, we have (with obvious notations) thanks to the reverse triangle inequality,

$$|\Psi(y) - \Psi(x)| \leq \|y - x\|_E \leq \left\| \sum_{i=1}^{p} (y_i - x_i) e_i \right\|_E \leq \left(\sum_{i=1}^{p} \|e_i\|_E \right) \|y - x\|_E^\infty.$$

Consequently, the map Ψ is continuous on E. Note in passing that the previous calculation shows that the norm $\|\cdot\|_E^\infty$ is stronger than $\|\cdot\|_E$ (take $y=0$ to see this).

Knowing that by the first step the unit sphere of E for the norm $\|\cdot\|_E^\infty$ is compact, we deduce that Ψ restricted to S is bounded and attains its bounds. This means in particular that there exists an x_0 such that

$$\|x_0\|_E^\infty = 1 \quad \text{and} \quad \|x\|_E \geq \|x_0\|_E \quad \text{for all} \quad x \in E \quad \text{such that} \quad \|x\|_E^\infty = 1.$$

By homogeneity, we conclude that

$$\forall x \in E, \|x\|_E \geq \|x_0\|_E \|x\|_E^\infty.$$

Finally, it is clear that x_0 cannot be zero. So $\|x_0\|_E > 0$. Consequently, the norm $\|\cdot\|_E$ is stronger than $\|\cdot\|_\infty$.

Finally, the two norms are therefore equivalent.

Third step: End of the proof of compactness. Knowing that every norm of E is equivalent to $\|\cdot\|_E^\infty$, the first step now allows us to conclude that every closed bounded set of E is compact. ■

Corollary 4.5.1

All finite-dimensional normed vector spaces are complete.

Proof. Let E be a finite-dimensional normed vector space. Since all norms on E are equivalent we can, without loss of generality, assume that E is equipped with the norm $\|\cdot\|_E^\infty$. With this choice of norm, it clearly appears that a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in E if and only if the sequences of its components $(x_n^i)_{n\in\mathbb{N}}$ for $i \in \{1, \ldots, p\}$ are Cauchy in $\mathbb{K}$. Since $\mathbb{K}$ is complete, we deduce that the sequences of the components converge, and then that $(x_n)_{n\in\mathbb{N}}$ also converges. ■

In infinite dimension, vector subspaces are not necessarily closed. For example, if we consider the vector space c_0 of real sequences tending to 0 at infinity, equipped with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, and the vector subspace F of c_0 consisting of real sequences that are zero from some rank on, then F is a dense vector subspace of c_0 that is not closed.

We however have the following result:

Theorem 4.5.2

Let $(E, \|\cdot\|_E)$ be an arbitrary normed vector space and F a finite-dimensional vector subspace of E. Then F is closed in E.

Proof. Fix a basis $(e^1, \ldots, e^p)$ of F. Since all norms are equivalent in finite dimension, the norm of E restricted to F is equivalent to the norm $\|\cdot\|_\infty$ on F associated to $(e^1, \ldots, e^p)$.

Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in F. It is thus Cauchy in the sense of the norm $\|\cdot\|_\infty$ introduced above and it is then immediate that all the components of the sequence with respect to $(e^1, \ldots, e^p)$ are Cauchy sequences in $\mathbb{K}$, hence convergent. ■

Theorem 4.5.3

The closed unit ball of the normed vector space E is compact if and only if E is finite-dimensional.

Proof. Only the direct implication remains to be proven. We reason by contraposition: assume E is infinite-dimensional. We can then construct by recurrence a sequence $(e_k)_{k\in\mathbb{N}}$ of linearly independent vectors. The sequence $(V_k)_{k\in\mathbb{N}}$ of vector subspaces defined by $V_k \overset{\text{def}}{=} \text{Vect}(e_0, \ldots, e_k)$ is a strictly increasing sequence of closed vector subspaces distinct from E. By applying Riesz's theorem, we can then construct by recurrence a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors such that $d(x_n, V_{n-1}) \geq 1/2$ and $x_n \in V_n$. We visibly have $\|x_n - x_m\| \geq 1/2$ for $n \neq m$. Consequently the sequence $(x_n)_{n\in\mathbb{N}}$ has no cluster point and the ball $B_E(0,1)$ is therefore not compact. ■

Corollary 4.5.2

Let E be a finite-dimensional normed vector space. Then

1. every bounded sequence in E admits a cluster point,
2. every bounded sequence having a unique cluster point converges.

Example 4.7. *Consider the space $E = \mathcal{C}([0,\pi];\mathbb{R})$ equipped with the norm*

$$\|f\|_{L^2} \overset{def}{=} \sqrt{\frac{1}{\pi}\int_0^\pi |f(t)|^2 dt}.$$

For $n \in \mathbb{N}^*$, set $f_n(x) = \sin nx$. It is clear that $\|f_n\|_{L^2}^2 = 1/2$ and that $\|f_n - f_m\|_{L^2} = 1$ for $n \neq m$. So the sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence in $B_E(0,1)$ that has no cluster point. We conclude that E is infinite-dimensional.

Remark 4.6. *In a normed vector space $(E, \|\cdot\|_E)$ of infinite dimension, all compacts have empty interior. Indeed if the set K does not have empty interior then it contains a closed ball $\overline{B}_E(x_0, r)$ with $r > 0$. If K were compact then the closed ball $\overline{B}_E(x_0, r)$ would also be compact and hence E, of finite dimension.*

Theorem 4.5.4

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces, and $f \in L(E;F)$ a linear map. If E is finite-dimensional then f is necessarily continuous.

Proof. Fix a basis $(e^1, \cdots, e^p)$ of E. Since on E all norms are equivalent, we can assume that $\|x\|_E = \max_{1\leq i\leq p}|x^i|$ where $(x^1, \cdots, x^p)$ are the coordinates of x with respect to $(e^1, \cdots, e^p)$.

We thus have, by the triangle inequality and the linearity of f,

$$\|f(x)\|_F = \|\sum_{i=1}^{p} x^i f(e^i)\|_F \leq \left(\sum_{i=1}^{p}\|f(e^i)\|_F\right)\|x\|_E,$$

whence the continuity of f. ■

The reader may verify that more generally we have the following result:

Theorem 4.5.5

Suppose that $E_1, \cdots, E_p$ are finite-dimensional. Then every p-linear map from $E_1 \times \cdots \times E_p$ to F is continuous.

§4.6 Exercices

Exercice 6. *Let $E = \mathcal{C}([0,1],\mathbb{R})$. For $f \in E$, we give*

$$\|f\|_1 = \int_0^1 |f(t)|dt,$$

and let ϕ be the endomorphism of E defined by

$$\phi(f)(x) = \int_0^x f(t)dt.$$

1. *Show that $\|\cdot\|_1$ is a norm on E.*
2. *Justify the sentence: "ϕ is an endomorphism of E".*
3. *Demonstrate that ϕ is continuous.*
4. *For $n \geq 0$, we consider f_n the elements of E defined by $f_n(x) = ne^{-nx}, x \in [0,1]$. Calculate $\|f_n\|_1$ and $\|\phi(f_n)\|_1$.*
5. *We put $\phi = \sup_{f\neq 0_E}\frac{\|\phi(f)\|_1}{\|f\|_1}$. Determine ϕ.*

Exercice 7. *Let $E = \mathcal{C}^\infty([0,1],\mathbb{R})$. We consider the operator of differentiation*

$$\begin{aligned} D : E &\to E \\ f &\mapsto f'. \end{aligned}$$

Show that, regardless of the norm N that we endow E with, D is never a continuous linear application from (E,N) to (E,N).

Exercice 8. *Let E be a normed vector space and $u \in \mathcal{L}(E)$. Prove that u is continuous if and only if $\{x \in E; \|u(x)\| = 1\}$ is closed.*

Exercice 9. *Let E be a normed vector space and u an endomorphism of E such that for all $x \in E, \|u(x)\| \leq \|x\|$. For all $n \in \mathbb{N}$, we define*

$$v_n = \frac{1}{n+1} \sum_{k=0}^{n} u^k.$$

1. *Simplify $v_n \circ (u - Id)$.*
2. *Show that $\ker(u - Id) \cap Im(u - Id) = \{0\}$.*
3. *Now suppose that E is finite-dimensional. Prove that*

$$\ker(u - Id) \oplus Im(u - Id) = E.$$

4. *Let p be the projection onto $\ker(u - Id)$ parallel to $Im(u - Id)$. Prove that for all $x \in E, v_n(x) \to p(x)$.*

Chapter 5

Connectedness and Convexity

§5.1 Connectedness

The notion of connectedness is very important in topology. If one seeks to have an intuitive representation of it, one will remember that a connected subset consists of ”only one piece”.

§5.1.1 General Framework

The mathematical definition of connectedness that we give below may seem at first glance rather far from the intuitive representation we have tried to impose on the reader:

Proposition 5.1.1

Let $(X, \mathcal{O})$ be a topological space and A a subset of X. The following three properties are equivalent:

1. A and $\emptyset$ are the only open and closed subsets of A,
2. there is no pair of nonempty open subsets of A, disjoint and whose union equals A,
3. there is no pair of closed subsets of A nonempty, disjoint and whose union equals A.

If one of these three properties is verified, we say that A is a **connected** subset of X.

Proof. (i) $\Rightarrow$ (ii): Let Ω_1 and Ω_2 be two disjoint open sets whose union equals A. Then $\Omega_2 = A \setminus \Omega_1$ so Ω_2 is both open and closed. We deduce that Ω_2 is either $\emptyset$ or A. (ii) $\Rightarrow$ (i): Let Ω be an open and closed subset of A. Then the same is true for $A \setminus \Omega$, so one of the two sets Ω and $A \setminus \Omega$ must be empty. (ii) $\iff$ (iii): It suffices to take complements. ■

Example 5.1.
1. *Let $(X, \mathcal{O})$ be a Hausdorff topological space and A a finite subset of X having at least two elements. It is easy to see that all the singletons of A are both open and closed. Consequently A is not connected.*
2. *Every normed vector space is connected.*

Proposition 5.1.2

The image of a connected subset by a continuous map is connected.

Proof. Let $(X, \mathcal{O})$ and $(X', \mathcal{O}')$ be two topological spaces, A a connected subset of X and $f \in C(A; X')$. Denote $B = f(A)$. Let U be an open and closed subset of B (for the induced topology). By continuity of f, the subset $f^{-1}(U)$ is open and closed in A. Since A is connected, we therefore have $f^{-1}(U) = \emptyset$ (in which case $U = \emptyset$) or else $f^{-1}(U) = A$ (and then $U = B$). ∎

Corollary 5.1.1

A subset A of the topological space $(X, \mathcal{O})$ is connected if and only if every continuous map from A to $\{0, 1\}$ equipped with the discrete topology is constant.

Proof. Let A be connected and f a continuous map from A to $\{0, 1\}$. Then $f(A)$ must be a connected subset of $\{0, 1\}$. Knowing that $\{0, 1\}$ is not connected, this means that $f(A)$ must consist of only one element.

To show the converse, let us proceed by contraposition. Suppose therefore that A is not connected. Then there exist two closed sets B and C nonempty, disjoint and such that $A = B \cup C$. We define the function f on A by $f(x) = 0$ if $x \in B$ and $f(x) = 1$ if $x \in C$. By construction of f, one easily verifies that for every closed set F of $\{0, 1\}$, the set $f^{-1}(F)$ is a closed set of A. So f is continuous and nonconstant. ∎

Corollary 5.1.2

The union of a family of connected subsets with nonempty intersection is connected.

Proof. Let $(A_i)_{i \in I}$ be a family of connected sets with intersection B nonempty and f a continuous map from $\bigcup_{i \in I} A_i$ to $\{0, 1\}$. By connectedness of A_i, f is constant on each A_i. If we denote by a_i the value of this constant, we deduce that f restricted to B (which is not empty) must be constant and equal to a_i for all i. Consequently, all the a_i are equal to each other, and f is therefore constant on $\bigcup_{i \in I} A_i$. We conclude thanks to the previous corollary. ∎

Remark 5.1. *The union of an arbitrary family of connected sets is not connected in general. Similarly, the intersection of two connected sets is not always connected (in $\mathbb{R}^2$, consider for example the intersection of an annulus and a rectangle).*

Corollary 5.1.3

Let A be a connected subset of $(X, \mathcal{O})$. Then every subset B of X such that $A \subset B \subset \overline{A}$ is connected.

Proof. Let f be a continuous map from B to $\{0, 1\}$ and b a point of B. Then f restricted to A is constant. Suppose for example that f is 0 on A. Since $b \in \overline{A}$, we have

$$f(b) = \lim_{\substack{a \to b \\ a \in A}} f(a) = 0.$$

We conclude that f is zero on B. So B is connected. ■

§5.1.2 Connected Subsets of $\mathbb{R}$

Theorem 5.1.1

The connected subsets of $\mathbb{R}$ are the intervals.

Proof. Consider first the case of a closed bounded interval of $\mathbb{R}$: $I = [a, b]$. Let F_1 and F_2 be two disjoint closed subsets of $[a, b]$. Since $[a, b]$ is closed in $\mathbb{R}$, F_1 and F_2 are in fact two closed sets of $\mathbb{R}$, and since they are bounded, they are compact. Suppose by contradiction that these two compacts are nonempty. Since they are disjoint, we have $d(F_1, F_2) > 0$, and there exist $x_1 \in F_1$ and $x_2 \in F_2$ such that $d(F_1, F_2) = |x_1 - x_2|$ (exercise: prove it). The point $(x_1 + x_2)/2$ also belongs to the interval $[a, b]$ and must therefore belong to one of the two closed sets, for example F_1. But

$$d\left(\frac{x_1 + x_2}{2}, x_2\right) = \left|\frac{x_1 - x_2}{2}\right| = \frac{d(F_1, F_2)}{2},$$

which is absurd. So F_1 or F_2 is empty and $[a, b]$ is indeed connected.

Since every interval of $\mathbb{R}$ is an increasing union of closed bounded intervals, Corollary 2 of page 38 allows us to conclude that every interval of $\mathbb{R}$ is connected.

If $A \subset \mathbb{R}$ is not an interval, there exist two points x and y of A such that $x < y$ and $[x, y] \not\subset A$. Taking $y_0 \in [x, y] \setminus A$, we observe that $A \cap]-\infty, y_0[$ and $A \cap [y_0, +\infty[$ are two nonempty disjoint closed subsets of A whose union equals A. So A is not connected. ■

Theorem 5.1.2: Intermediate value theorem

Let A be a connected subset of a topological space $(X, \mathcal{O})$, and f a continuous map from A to $\mathbb{R}$. Then $f(A)$ is an interval of $\mathbb{R}$.

Proof. We know that $f(A)$ is a connected subset of $\mathbb{R}$. It only remains to apply the previous theorem. ■

Corollary 5.1.4

Let f be a continuous function from A to $\mathbb{R}$ with A both compact and connected. Then there exists $(a, b) \in \mathbb{R}^2$ with $a \leq b$ such that $f(A) = [a, b]$.

Proof. By the intermediate value theorem, the set $f(A)$ is an interval. Moreover, by compactness of A, the set $f(A)$ must be a compact subset of $\mathbb{R}$. Consequently, it is a closed bounded interval. ■

§5.1.3 Arcwise Connectedness

Definition 5.1.1

Let $(X, \mathcal{O})$ be a topological space, and A a nonempty subset of X. Let (a, b) be a pair of points of A. We say that a map φ defined on $[0, 1]$ is a **path** in A from a to b if it is continuous from $[0, 1]$ to A, and satisfies $\varphi(0) = a$ and $\varphi(1) = b$.

Remark 5.2. *The image of* $[0, 1]$ *by* φ *is a continuous curve with endpoints* a *and* b*.*

Example 5.2. *Let* E *be a vector space equipped with a topology for which vector addition and scalar multiplication are continuous operations (we speak of a topological vector space). To every closed segment* $[a, b] = \{(1 - t)a + tb, t \in [0, 1]\}$ *of* E*, we can associate a path from* a *to* b*. It suffices to consider* $\varphi : t \mapsto (1-t)a+tb$*.*

Definition 5.1.2

We say that the subset A of X is **arcwise connected** if for every pair (a, b) of points of A there exists a path in A from a to b.

Proposition 5.1.3

Every arcwise connected subset is connected.

Proof. We will use the characterization of connectedness given by Corollary 1. So let A be arcwise connected and $f \in C(A; \{0, 1\})$. Let a and b be two arbitrary points of A. There exists a path $\varphi \in C([0, 1]; A)$ such that $\varphi(0) = a$, $\varphi(1) = b$. The map $f \circ \varphi$ is continuous from $[0, 1]$ (a connected subset of $\mathbb{R}$) to $\{0, 1\}$ and is therefore constant. In particular

$$f(a) = f \circ \varphi(0) = f \circ \varphi(1) = f(b).$$

So f is constant. ■

Remark 5.3. *The converse is, in general, false.*

We however have the following result:

Theorem 5.1.3

For open subsets of normed vector spaces, connectedness is equivalent to arcwise connectedness.

Proof. Let A be a nonempty connected open subset of E. Fix $a \in A$ and consider the set

$$B \stackrel{\text{def}}{=} \{b \in A \mid \text{there exists a path in } A \text{ joining } a \text{ and } b\}.$$

By definition itself, the set B is arcwise connected. It remains to show that $B = A$.

- B is not empty because it contains the point a.
- The set B is open. Indeed, if $b \in B$, there exists a path in A from a to b and, since A is open, a nonempty ball $B(b, r)$ included in A. For every point c of $B(b, r)$, the segment $[b, c]$ is included in $B(b, r)$ and hence in A. There therefore exists a path in A from b to c. Finally, the union of a path from a to b with a path from b to c is a path from a to c. We therefore conclude that $c \in B$, and then that $B(b, r) \subseteq B$.
- The set B is closed. Let $b \in \overline{B} \cap A$. Choose $r > 0$ such that $B(b, r) \subseteq A$. Since $b \in \overline{B}$, the set $B(b, r) \cap B$ is not empty and therefore contains a point c. By definition of the set B, there exists a path in A joining a to c. Moreover, the segment $[b, c]$ belongs to $B(b, r)$ and is thus included in A, and we finally deduce that $b \in B$. Since the set A is connected, we conclude that $B = A$.

■

§5.2 A Bit of Convexity

Definition 5.2.1

A subset A of a vector space E is said to be **convex** if

$$\forall (x, y) \in A \times A,\ [x, y] \subseteq A.$$

Remark 5.4. *The notions of convexity and arcwise connectedness are topological notions (i.e., they depend only on the chosen topology). In the case of a normed vector space, they are therefore invariant under changing the norm to an equivalent norm. The notion of convexity is an algebraic notion and is therefore independent of the chosen topology.*

Proposition 5.2.1

In a topological vector space, every convex set is arcwise connected.

Proof. Let A be a convex subset of E. Let $(x, y) \in A^2$. By convexity, the image of the continuous map

$$\begin{cases} [0,1] & \longrightarrow E \\ t & \longmapsto tx + (1-t)y \end{cases}$$

is included in A. So A is indeed arcwise connected. ■

Proposition 5.2.2

Let A be convex. Then for every n-tuple $(x_1, \cdots, x_n)$ of elements of A and every n-tuple $(\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ satisfying $\alpha_1 + \cdots + \alpha_n = 1$, we have

$$\sum_{i=1}^{n} \alpha_i x_i \in A.$$

Proof. We reason by induction on n. The case $n = 1$ is evident. Assume the result established for every $(x_1, \cdots, x_n) \in E^n$ and $(\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ satisfying $\alpha_1 + \cdots + \alpha_n = 1$. Let $(x_1, \cdots, x_{n+1}) \in A^{n+1}$ and $(\beta_1, \cdots, \beta_{n+1}) \in [0,1]^{n+1}$ such that $\beta_1 + \cdots + \beta_{n+1} = 1$. We can moreover assume that all coefficients β_i are in $[0,1]$ (otherwise the result is contained in the induction hypothesis). Let $x = \sum_{i=1}^{n+1} \beta_i x_i$. We observe that

$$x = \beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) y \quad \text{with} \quad y = \sum_{i=1}^{n} \frac{\beta_i}{1 - \beta_{n+1}} x_i.$$

Since $\sum_{i=1}^{n} \frac{\beta_i}{1-\beta_{n+1}} = 1$, the induction hypothesis ensures that $y \in A$. By convexity of A, we therefore have $x \in A$ as desired. ■

We leave it to the reader to verify the following properties:

1. The image of a convex subset by a linear map is convex.
2. The preimage of a convex set by a linear map is convex.
3. The intersection of an arbitrary family of convex sets is convex.
4. Let A and B be two convex sets, and $(\alpha, \beta) \in \mathbb{R}^2$. Then $\alpha A + \beta B$ is convex.

Definition 5.2.2

Let A be an arbitrary subset of E. The **convex hull** of A is the smallest convex set containing A.

Proposition 5.2.3

Let A be a subset of E. The convex hull of A is the set of finite linear combinations with positive coefficients whose sum is 1, of elements of A.

Proof. Denote by B the set of finite linear combinations with positive coefficients whose sum is 1, of elements of A. It is clear that this set is convex. Moreover, if C is another convex set containing A, it must in particular contain all convex combinations of elements of A, so B. ■

Proposition 5.2.4

In a normed vector space, the closure and the interior of a convex set are convex.

Proof. Let A be a nonempty convex set, $(x, y) \in \overline{A}^2$ and $t \in [0, 1]$. Then there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of A such that

$$\lim_{n\to+\infty} x_n = x \quad \text{and} \quad \lim_{n\to+\infty} y_n = y.$$

By convexity of A, we have $tx_n + (1-t)y_n \in A$ for all $n \in \mathbb{N}$. Moreover it is clear that

$$\lim_{n\to+\infty} tx_n + (1-t)y_n = tx + (1-t)y$$

so $tx + (1-t)y \in \overline{A}$. This shows that $\overline{A}$ is convex.

Let us now show that $\hat{A}$ is also convex. We exclude the case $\hat{A} = \emptyset$ which is trivial. Let $(x, y) \in \hat{A}^2$. Then there exists $r > 0$ such that the open balls $B(x, r)$ and $B(y, r)$ are included in A. By convexity of A, we therefore have $tB(x, r) + (1-t)B(y, r) \subset A$ for all $t \in [0, 1]$. One easily establishes that

$$tB(x, r) + (1-t)B(y, r) = B(tx + (1-t)y, r).$$

So $tx + (1-t)y \in \hat{A}$, and $\hat{A}$ is indeed convex. ■

Definition 5.2.3

Let A be a convex subset of E and $f : A \to \mathbb{R}$. We say that f is **convex** if

$$\forall \alpha \in [0, 1], \quad \forall (x, y) \in A^2, \quad f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Proposition 5.2.5

Let $f : A \to \mathbb{R}$ be a convex function. Then for every real number c the sets $f^{-1}(]-\infty, c])$ and $f^{-1}(]-\infty, c[)$ are convex.

Proof. Suppose $f^{-1}(]-\infty, c])$ is nonempty and give us x and y two elements of $f^{-1}(]-\infty, c])$ and $t \in [0,1]$. By convexity of f, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq tc + (1-t)c = c,$$

whence the result. The proof of the convexity of $f^{-1}(]-\infty, c[)$ is similar. ■

Example 5.3. *Let $f : E \to \mathbb{R}$ be an affine function, and $a \in \mathbb{R}$. Then the sets $f^{-1}([a, +\infty[)$, $f^{-1}(]a, +\infty[)$, $f^{-1}(]-\infty, a])$ and $f^{-1}(]-\infty, a[)$ are convex.*

Proposition 5.2.6

If $f : A \to \mathbb{R}$ is a convex function then for every $(\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ such that $\sum_{i=1}^n \alpha_i = 1$, we have

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i).$$

Theorem 5.2.1

Let E be a finite-dimensional normed vector space and U an open convex subset of E. Then every convex function defined on U is continuous.

Proof. Let $f : U \to \mathbb{R}$ be a convex function. We fix a basis $(e_1, \cdots, e_n)$ of E and we equip E with the norm $\|x\| = \sum_{i=1}^n |x_i|$ for $x = \sum_{i=1}^n x_i e_i$ (this does not affect the continuity properties of f since in finite dimension all norms are equivalent). Fix a $b \in U$. The goal is to show the continuity of f at b.

1. **Reduction to the case $b = 0$, $f(0) = 0$ and $\overline{B}(0,1) \subset U$.**

   By considering the function $g : x \mapsto f(\lambda x + b) - f(b)$ with $\lambda > 0$ such that $\overline{B}(b, \lambda) \subset U$, we can reduce to studying continuity at 0 for a convex function defined on an open set containing the closed unit ball $\overline{B}(0,1)$ and vanishing at 0. Note indeed that f and g are simultaneously convex and that f is continuous at b if and only if g is continuous at 0.

2. **Bounding g on $\overline{B}(0,1)$.**

   Denote a_0 the origin, $a_i^+ = e_i$ and $a_i^- = -e_i$ for $i = 1, \cdots, n$. We remark that every point $x = \sum_{i=1}^n x_i e_i$ of $\overline{B}(0,1)$ decomposes as

$$x = (1 - \|x\|)a_0 + \sum_{i=1}^n |x_i| a_i^{\varepsilon_i},$$

$$\varepsilon_i = \begin{cases} +, & \text{if } x_i \geq 0, \\ -, & \text{otherwise.} \end{cases}$$

It is clear that $1 - \|x\| \in [0,1], |x_i| \in [0,1]$ for $i \in \{1, \cdots, n\}$ and $1 - \|x\| + \sum_{i=1}^{n} |x_i| = 1$. By convexity of g and since $g(a_0) = 0$, we therefore have, by the previous proposition,

$$\begin{aligned} g(x) &\leq M\|x\|, \\ M &= \max\{g(a_1^+), \cdots, g(a_n^+), g(a_1^-), \cdots, g(a_n^-)\}. \end{aligned} \tag{4.1}$$

3. **End of the proof.**
   Writing that $0 = \frac{x}{2} + \frac{(-x)}{2}$ and using $g(0) = 0$, we obtain

   $$g(x) \geq -g(-x) \geq -M\|x\|.$$

   We conclude that

   $$\forall x \in \overline{B}(0,1), |g(x)| \leq M\|x\|.$$

   So g is continuous at 0, and f, at b.

■

Remark 5.5. *The above result is false in general if we do not assume that U is open or if we are in infinite dimension.*

§5.3 Exercices

Exercice 10. *Let E be a normed vector space and A, B two arc-connected subsets of E.*

1. *Prove that $A \times B$ is arc-connected.*
2. *Deduce that $A + B$ is arc-connected.*
3. *Is the interior of A always arc-connected?*

Exercice 11. *Let $(A_i)_{i \in I}$ be a family of arc-connected subsets of the normed vector space E such that $\bigcap_{i \in I} A_i \neq \varnothing$. Prove that $\bigcup_{i \in I} A_i$ is arc-connected.*

Exercice 12. *Let $\phi, \psi : \mathbb{C} \to \mathbb{C}$ be continuous functions such that, for all $z \in \mathbb{C}$, $\exp(\phi(z)) = \exp(\psi(z))$. Prove that there exists $k \in \mathbb{Z}$ such that, for all $z \in \mathbb{C}$, $\phi(z) = \psi(z) + 2ik\pi$.*

Exercice 13. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. We want to demonstrate using arc-connectedness the classical result: if f is continuous and injective, then f is strictly monotonic. To do this, we define $C = \{(x,y) \in \mathbb{R}^2; x > y\}$ and $F(x,y) = f(x) - f(y)$ for $(x,y) \in C$.*

1. *Prove that $F(C)$ is an interval.*
2. *Conclude.*

Exercice 14. *We say that two subsets A and B of two normed vector spaces E and F are homeomorphic if there exists a bijection $f : A \to B$ such that both f and f^{-1} are continuous.*

1. *Prove that $\mathbb{R}^2 \backslash \{0\}$ is arc-connected.*
2. *Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.*
3. *Prove that $[0, 1]$ and the unit circle are not homeomorphic.*

Exercice 15. *Let E be a normed vector space of dimension two or higher (possibly infinite-dimensional). Prove that its unit sphere $\mathcal{S}_E$ is arc-connected.*

Exercice 16. *Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a differentiable function. We denote $A = \{(x, y) \in I \times I; x < y\}$.*

1. *Prove that A is an arc-connected subset of $\mathbb{R}^2$.*
2. *For $(x, y) \in A$, define $g(x, y) = \frac{f(y)-f(x)}{y-x}$. Prove that $g(A) \subset f'(I) \subset \overline{g(A)}$.*
3. *Prove that $f'(I)$ is an interval.*

Exercice 17. *Let E be a finite-dimensional normed vector space and let $p : [0, 1] \to \mathcal{L}(E)$ be a continuous function such that for all $t \in [0, 1]$, $p(t)$ is a projection. Prove that the map $t \mapsto \mathrm{rg}(p(t))$ is constant.*

Exercice 18. *Let A be a subset of a normed vector space E, and let $f : A \to F$ be a continuous function, where F is a normed vector space. We say that f is locally constant if, for every $a \in A$, there exists $r > 0$ such that f is constant on $B(a, r) \cap A$. The goal of this exercise is to prove that if A is arc-connected and f is locally constant, then f is constant. To do this, we fix $a, b \in A$ and consider a continuous path $\phi : [0, 1] \to A$ such that $\phi(0) = a$ and $\phi(1) = b$. We define $t = \sup\{s \in [0, 1]; f(\phi(s)) = f(a)\}$.*

1. *Prove that $t = 1$.*
2. *Conclude.*

Exercice 19. *Let A be an arc-connected subset of a normed vector space, and let B be a subset of A that is both open and closed relative to A. Define the function $f : A \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

1. *Prove that f is continuous.*
2. *Deduce that $B = \varnothing$ or $B = A$.*

Exercice 20.

1. *Prove that the arc-connected components of an open subset of $\mathbb{R}^n$ are open.*

2. *Deduce that any open subset of $\mathbb{R}$ is a union of pairwise disjoint open intervals.*

3. *Prove that this union is either finite or countably infinite.*

Exercice 21. *Determine all continuous functions $f : \mathbb{C} \to \mathbb{C}$ such that, for all $z \in \mathbb{C}$, $(f(z))^2 = z^2$.*

Exercice 22. *Let E be a normed vector space, C a convex subset of E, and D a set such that $C \subset D \subset \bar{C}$. Prove that D is arc-connected.*

Exercice 23. *Let A, B be two subsets of a normed vector space E. Are the following assertions true or false?*

1. *If A is connected, then its boundary is connected.*

2. *If $\bar{A}$ is connected, then A is connected.*

3. *If A and B are connected, then $A \cap B$ is connected.*

4. *If A and B are convex, then $A \cap B$ is connected.*

5. *If A and B are connected, then $A \cup B$ is connected.*

6. *If $f : A \to F$ is continuous, with A convex and F a normed vector space, then $f(A)$ is convex.*

Chapter 6

The Great Theorems of Functional Analysis

§6.1 The Baire Theorem

Theorem 6.1.1: Baire, first form

Let (X, d) be a complete metric space and $(F_n)_{n\in\mathbb{N}}$ a sequence of closed subsets of X, with empty interior. Then $\bigcup_{n\in\mathbb{N}} F_n$ has empty interior.

Theorem 6.1.2: Baire, second form

Let (X, d) be a complete metric space and $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}} \Omega_n$ is dense in X.

Proof. Let us show the second form. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of dense open sets and V a nonempty open set of X. The goal is to show that $V \cap (\bigcap_{n\in\mathbb{N}} \Omega_n) \neq \emptyset$.

1. By density of Ω_0, the open set $\Omega_0 \cap V$ is nonempty. So there exists $x_0 \in X$ and $r_0 > 0$ such that $\overline{B}(x_0, r_0) \subset \Omega_0 \cap V$.
2. By recurrence, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and a sequence $(r_n)_{n\in\mathbb{N}}$ of strictly positive real numbers such that for every $n \in \mathbb{N}$:
   - $\overline{B}(x_{n+1}, r_{n+1}) \subset \Omega_{n+1} \cap B(x_n, r_n)$
   - $r_{n+1} \leq \frac{r_n}{2}$
3. The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy because for $m > n$:
$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} r_k \leq r_n \sum_{k=0}^{\infty} \frac{1}{2^k} = 2r_n$$
   and $r_n \to 0$.
4. Since X is complete, (x_n) converges to a point $x \in X$. By construction, $x \in \overline{B}(x_n, r_n)$ for every n, so $x \in V \cap (\bigcap_{n\in\mathbb{N}} \Omega_n)$.

■

Definition 6.1.1

We say that a Hausdorff topological space $(X, \mathcal{O})$ has the **Baire property** if for every sequence $(\Omega_n)_{n\in\mathbb{N}}$ of dense open subsets of X, the set $\bigcap_{n\in\mathbb{N}} \Omega_n$ is dense.

Proposition 6.1.1

Let X be a Hausdorff topological space having the Baire property and Ω an open subset of X. Then Ω equipped with the topology induced by X also has the Baire property.

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of Ω. Since Ω is open, one verifies that $(\Omega_n \cup (X \setminus \overline{\Omega}))_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X. By the Baire property of X, $\bigcap_{n\in\mathbb{N}}(\Omega_n \cup (X \setminus \overline{\Omega}))$ is dense in X.

Let V be an open subset of Ω. Then V is an open subset of X and:

$$V \cap \left(\bigcap_{n\in\mathbb{N}} (\Omega_n \cup (X \setminus \overline{\Omega})) \right) = V \cap \left(\bigcap_{n\in\mathbb{N}} \Omega_n \right)$$

is nonempty, so $\bigcap_{n\in\mathbb{N}} \Omega_n$ is dense in Ω. ■

§6.1.1 The Banach-Steinhaus Theorem

Theorem 6.1.3: Banach-Steinhaus

Let E be a Banach space, F a normed vector space and $(T_i)_{i\in I}$ a family of maps in $\mathcal{L}(E;F)$. Assume that for every $x \in E$, the set $\{T_i(x) | i \in I\}$ is bounded in F. Then there exists $M \geq 0$ such that $\forall i \in I$, $\|T_i\|_{\mathcal{L}(E;F)} \leq M$.

Proof. For every $n \in \mathbb{N}$, define:

$$A_n = \{x \in E | \forall i \in I, \|T_i(x)\|_F \leq n\}$$

Each A_n is closed (as an intersection of the closed sets $T_i^{-1}(\overline{B}_F(0,n))$) and we have $\bigcup_{n\in\mathbb{N}} A_n = E$.

By the Baire theorem, there exists $n_0 \in \mathbb{N}$ such that A_{n_0} has a nonempty interior. Let $x_0 \in E$ and $r_0 > 0$ such that $\overline{B}(x_0, r_0) \subset A_{n_0}$.

Then for every $i \in I$ and every $y \in \overline{B}_E(0,1)$, we have:

$$\|T_i(y)\|_F = \frac{1}{r_0}\|T_i(r_0 y)\|_F \leq \frac{1}{r_0}(\|T_i(x_0 + r_0 y)\|_F + \|T_i(x_0)\|_F) \leq \frac{2n_0}{r_0}$$

So $\|T_i\|_{\mathcal{L}(E;F)} \leq \frac{2n_0}{r_0}$ for every $i \in I$. ■

Corollary 6.1.1

Let E be a Banach space, F a normed vector space, and $(T_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}(E;F)$. Assume that for every $x \in E$, the sequence $(T_n(x))_{n\in\mathbb{N}}$ converges in F to a limit denoted $T(x)$. Then $T \in \mathcal{L}(E;F)$.

Proof. The linearity of T follows from the linearity of the T_n. By the Banach-Steinhaus theorem, there exists $M \geq 0$ such that $\|T_n\| \leq M$ for every n. Then for every $x \in E$:

$$\|T(x)\|_F = \lim_{n\to\infty} \|T_n(x)\|_F \leq M\|x\|_E$$

so T is continuous. ∎

§6.1.2 The Open Mapping Theorem

Definition 6.1.2

Let $(X, \mathcal{O}), (X', \mathcal{O}')$ be two topological spaces. We say that $f : X \to X'$ is an **open map** if the image of every open subset of X by f is an open subset of X'.

Theorem 6.1.4: Open mapping theorem

Let E and F be two Banach spaces, and $T \in \mathcal{L}(E;F)$ surjective. Then T is an open map.

Proof. The proof is done in several steps:

1. We show that there exists $c > 0$ such that $B_F(0,c) \subset \overline{T(B_E(0,1))}$.

   Indeed, let $X_n = T(B_E(0,n))$. Since T is surjective, $F = \bigcup_{n\in\mathbb{N}} X_n$. By the Baire theorem, there exists n_0 such that $\overline{X_{n_0}}$ has a nonempty interior. So there exists $y_0 \in F$ and $r > 0$ such that $B_F(y_0, r) \subset \overline{T(B_E(0,n_0))}$.

   By linearity and convexity, we deduce that $B_F(0,r) \subset \overline{T(B_E(0,2n_0))}$, and then by homogeneity that $B_F(0,c) \subset \overline{T(B_E(0,1))}$ with $c = \frac{r}{2n_0}$.

2. We show that $B_F(0,c) \subset T(B_E(0,1))$.

   Let $y \in B_F(0,c)$. We construct by recurrence a sequence (x_n) in E such that:

   - $\|x_n\|_E < 2^{-n}$
   - $\|y - T(x_1 + \cdots + x_n)\|_F < 2^{-n}c$

   The series $\sum x_n$ converges (since E is complete) to an element x with $\|x\|_E < 1$, and $T(x) = y$.

∎

Corollary 6.1.2

Let E and F be two Banach spaces, and $T \in \mathcal{L}(E;F)$ bijective. Then T^{-1} is continuous.

Proof. Since T is open, the preimage by T^{-1} of an open subset of E is an open subset of F, so T^{-1} is continuous. ■

§6.1.3 The Closed Graph Theorem

Definition 6.1.3

Let E, F be two normed vector spaces and T a linear map from E to F. The **graph** of T (denoted $G(T)$) is the following subset of $E \times F$:

$$G(T) = \{(x, y) \in E \times F \mid y = T(x)\}.$$

Theorem 6.1.5: Closed graph theorem

Let E and F be two Banach spaces, and T a linear map from E to F. Then T is continuous if and only if $G(T)$ is a closed subset of $E \times F$.

Proof. If T is continuous, then $G(T)$ is closed because if $(x_n, T(x_n)) \to (x, y)$, then by continuity $y = T(x)$.

Conversely, if $G(T)$ is closed, then it is a Banach space. The projection $\pi_E : G(T) \to E$ defined by $\pi_E(x, T(x)) = x$ is continuous, bijective and linear. By the open mapping theorem, π_E^{-1} is continuous. So $T = \pi_F \circ \pi_E^{-1}$ is continuous. ■

§6.2 Exercices

Exercice 24. *Let (E, d) be a complete metric space. We assume that E can be written as the countable union of closed sets:*

$$E = \bigcup_n F_n.$$

Then the set

$$\Omega = \bigcup_n \mathring{F}_n$$

is a dense open set in E.

Exercice 25. *Let I be a non-empty open interval in $\mathbb{R}$, and $f : T \longrightarrow \mathbb{R}$ be a differentiable function. Then the derivative function f' is continuous at every point of a dense subset of I.*

Exercice 26. *Consider* $E = C([0,1],\mathbb{C})$. *We have two normed spaces:* $X = (E, \|\cdot\|_1)$ *and* $Y = (E, \|\cdot\|_\infty)$, *where* $\|f\|_\infty = \sup |f(t)| / t \in [0,1]$ *and* $\|f\|_1 = \int_0^1 |f(t)| dt$. *Let's denote*

$$\begin{aligned} I_d : X &\to Y \\ x &\mapsto x \end{aligned}$$

as the identity function.

1. *Show that* I_d *is a bijective and continuous function. What is its norm?*

2. *Show that* I_d^{-1} *is not continuous.*

3. *Conclude.*

Exercice 27. *Let* E *and* F *be two normed vector spaces.*

1. *Show that if* E *is complete and* F *is a closed subspace of* E, *then the quotient space* $\left(E/F, |\cdot|_{E/F}\right)$ *is a Banach space.*

2. *Show that if* $T \in L(E,F)$, *then the application* $\tilde{T} : E/\ker(T) \to F$ *defined as* $\tilde{T}(x + \ker(T)) = T(x)$ *is a continuous function, and* $|\tilde{T}| = |T|$.

3. *Show that if* $T \in \mathcal{L}(E,F)$ *has finite rank and* $\ker(T)$ *is closed, then* T *is continuous.*

Exercice 28. *Let* X *be a Banach space,* Y *be a normed vector space, and* $T : X \to Y$ *be a continuous linear map. Suppose there exists a constant* $c > 0$ *such that* $|Tx| \geqslant c|x|$ *for all* $x \in X$. *Show that the image* $T(X)$ *is closed in* Y, *and* T *establishes an isomorphism between* X *and* $T(X)$.

Chapter 7

Hilbert Spaces

§7.1 Inner Product

Definition 7.1.1

Let E be a real vector space. A bilinear form $b : E \times E \to \mathbb{R}$ is an **inner product** if:

1. It is symmetric: $\forall x, y \in E, b(x, y) = b(y, x)$
2. It is positive definite: $\forall x \in E \setminus \{0\}, b(x, x) > 0$

Definition 7.1.2: Complex case

Let E be a complex vector space. A map $h : E \times E \to \mathbb{C}$ is a **Hermitian inner product** if:

1. It is linear with respect to the first variable
2. It is antilinear with respect to the second variable: $h(x, y + \lambda z) = h(x, y) + \overline{\lambda} h(x, z)$
3. It is Hermitian: $\forall x, y \in E, h(y, x) = \overline{h(x, y)}$
4. It is positive definite: $\forall x \in E \setminus \{0\}, h(x, x) > 0$

Theorem 7.1.1: Cauchy-Schwarz inequality

Let E be a pre-Hilbert space. Then for every $(x, y) \in E^2$, we have:

$$|(x \mid y)| \leq \|x\|\|y\|$$

with equality if and only if x and y are collinear.

Proof. In the real case, consider for $\lambda \in \mathbb{R}$:

$$f(\lambda) = \|x + \lambda y\|^2 = \|x\|^2 + 2\lambda(x|y) + \lambda^2\|y\|^2$$

This polynomial is always nonnegative, so its discriminant is non-positive:

$$\Delta = 4(x|y)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

whence the inequality.

In the complex case, write $(x|y) = |(x|y)|e^{i\theta}$ and consider $f(\rho) = \|x+\rho e^{i\theta}y\|^2$ for $\rho \in \mathbb{R}$. ■

Proposition 7.1.1: Important identities

In every pre-Hilbert space, we have:

1. **Parallelogram identity**:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

2. **Polarization identity**:
   - Real case: $(x|y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$
   - Complex case: $(x|y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$

§7.1.1 Orthogonality

Definition 7.1.3

We say that two elements x and y of a pre-Hilbert space E are **orthogonal** if $(x \mid y) = 0$. We denote $x \perp y$.

Theorem 7.1.2: Pythagorean theorem

If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Definition 7.1.4

For every nonempty subset A of a pre-Hilbert space E, we define the **orthogonal** of A by:

$$A^\perp = \{y \in E \mid \forall x \in A, (x \mid y) = 0\}$$

Proposition 7.1.2

For every $A \subset E$ nonempty, $A^\perp$ is a closed vector subspace of E.

Proof. Let (y_n) be a sequence in $A^\perp$ converging to y. For every $x \in A$:

$$|(x|y)| = \lim_{n\to\infty} |(x|y_n)| = 0$$

so $y \in A^\perp$. Stability under linear combinations is immediate. ■

Theorem 7.1.3: Gram-Schmidt orthonormalization process

Let $(a_1, \cdots, a_p)$ be a free family in a pre-Hilbert space E. Then there exists a unique orthonormal family $(e_1, \cdots, e_p)$ such that:

1. $\forall j \in \{1, \cdots, p\}$, $\mathrm{Vect}(e_1, \cdots, e_j) = \mathrm{Vect}(a_1, \cdots, a_j)$
2. $\forall j \in \{1, \cdots, p\}$, $(a_j \mid e_j) > 0$

Proof. We construct the family by recurrence:

- $e_1 = \frac{a_1}{\|a_1\|}$
- For $k \geq 1$: $e_{k+1} = \frac{a_{k+1} - \sum_{i=1}^{k} (a_{k+1}|e_i) e_i}{\|a_{k+1} - \sum_{i=1}^{k} (a_{k+1}|e_i) e_i\|}$

■

§7.1.2 Hilbert Spaces

Definition 7.1.5

A pre-Hilbert space complete for the norm associated to the inner product is called a **Hilbert space**.

Example 7.1. *1. $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with their usual inner product are Hilbert spaces.*

2. The space ℓ^2 of square-summable sequences: $(x|y) = \sum_{n=0}^{\infty} x_n \overline{y_n}$

3. The space $L^2([a,b])$ of square-integrable functions: $(f|g) = \int_a^b f(x)\overline{g(x)}dx$

Proposition 7.1.3

In a Hilbert space, every orthogonal series $\sum x_n$ such that $\sum \|x_n\|^2 < \infty$ is convergent and we have Parseval's equality:

$$\left\| \sum_{n=0}^{\infty} x_n \right\|^2 = \sum_{n=0}^{\infty} \|x_n\|^2$$

§7.1.3 Orthogonal Projections

Theorem 7.1.4

Let K be a nonempty closed convex subset of a Hilbert space H. Then for every $x \in H$, there exists a unique point $p_x \in K$ such that:

$$\|x - p_x\| = d(x, K) = \inf_{y \in K} \|x - y\|$$

Moreover, p_x is characterized by the condition:

$$\forall y \in K, \quad \operatorname{Re}(x - p_x \mid y - p_x) \leq 0$$

Proof. Let (y_n) be a sequence in K such that $\|x - y_n\| \to d(x, K)$. By the parallelogram identity:

$$\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4\left\|x - \frac{y_n + y_m}{2}\right\|^2$$

Since $\frac{y_n+y_m}{2} \in K$ (by convexity), we have $\left\|x - \frac{y_n+y_m}{2}\right\| \geq d(x, K)$, so (y_n) is Cauchy. Its limit p_x satisfies $\|x - p_x\| = d(x, K)$.

Uniqueness is also shown by the parallelogram identity. ■

Corollary 7.1.1

Let F be a closed vector subspace of H. For every $x \in H$, the projection p_x of x onto F is the unique point of F such that $x - p_x \in F^\perp$.

Proposition 7.1.4

Let F be a vector subspace of a Hilbert space H. The following assertions are equivalent:

1. F is closed
2. $H = F \oplus F^\perp$
3. $(F^\perp)^\perp = F$

§7.1.4 Riesz-Fréchet Representation Theorem

Theorem 7.1.5: Riesz-Fréchet representation theorem

Let H be a Hilbert space. Then for every $f \in H'$ (the topological dual of H), there exists a unique $x_f \in H$ such that:

$$\forall y \in H, \quad f(y) = (y \mid x_f)$$

Moreover, the map $f \mapsto x_f$ is an antilinear isometry: $\|x_f\| = \|f\|_{H'}$.

Proof. If $f = 0$, take $x_f = 0$. Otherwise, $\ker f$ is a closed hyperplane. Let $z \in (\ker f)^\perp$ with $\|z\| = 1$. Then for every $y \in H$:

$$y - \frac{f(y)}{f(z)} z \in \ker f$$

so $(y \mid z) = \frac{f(y)}{f(z)}$. Take $x_f = \overline{f(z)} z$.

The isometry comes from:

$$\|f\| = \sup_{\|y\|=1} |f(y)| = \sup_{\|y\|=1} |(y|x_f)| = \|x_f\|$$

by the Cauchy-Schwarz inequality. ■

§7.1.5 Lax-Milgram Theorem

Theorem 7.1.6: Lax-Milgram theorem

Let H be a Hilbert space and $a : H \times H \to \mathbb{K}$ a continuous and coercive sesquilinear form, i.e., there exists $c_0 > 0$ such that:

$$\forall x \in H, \quad \mathrm{Re}\, a(x, x) \geq c_0 \|x\|^2$$

Then for every $f \in H'$ there exists a unique $x \in H$ satisfying:

$$\forall y \in H, \quad a(y, x) = f(y)$$

Proof. By the Riesz theorem, there exists $u \in H$ such that $f(y) = (y|u)$. The map $A : H \to H$ defined by $(y|Ax) = a(y, x)$ is linear continuous and coercive. We then consider for $\rho > 0$ the map:

$$S_\rho(x) = x + \rho(u - Ax)$$

For ρ small enough, S_ρ is contractive. By the fixed point theorem, it admits a unique fixed point x which satisfies $Ax = u$. ■

§7.1.6 Hilbert Bases

Definition 7.1.6

A subset A of a Hilbert space H is said to be **total** if $\overline{\mathrm{Vect}(A)} = H$.

Proposition 7.1.5

A subset A of H is total if and only if $A^\perp = \{0\}$.

Definition 7.1.7

Let H be a Hilbert space. A sequence $(e_n)_{n\in\mathbb{N}}$ is a **Hilbert basis** of H if:

1. (e_n) is orthonormal: $(e_i|e_j) = \delta_{ij}$
2. (e_n) is total: $\overline{\mathrm{Vect}\{e_n, n \in \mathbb{N}\}} = H$

Theorem 7.1.7: Fourier series decomposition

Let H be a separable Hilbert space and (e_n) a Hilbert basis of H. Then for every $x \in H$, we have:

$$x = \sum_{n=0}^{\infty} (x|e_n)e_n \quad \text{and} \quad \|x\|^2 = \sum_{n=0}^{\infty} |(x|e_n)|^2 \quad \text{(Parseval's equality)}$$

Proof. Let $x_N = \sum_{n=0}^{N} (x|e_n)e_n$. For $M > N$:

$$\|x_M - x_N\|^2 = \sum_{n=N+1}^{M} |(x|e_n)|^2$$

The series $\sum |(x|e_n)|^2$ converges (since bounded by $\|x\|^2$), so (x_N) is Cauchy. Let y be its limit. Then $(x - y|e_n) = 0$ for every n, so $x = y$. ∎

§7.1.7 Weak Convergence

Definition 7.1.8

Let (x_n) be a sequence in a Hilbert space H and $x \in H$. We say that (x_n) **converges weakly** to x if:

$$\forall y \in H, \quad \lim_{n\to\infty} (y|x_n) = (y|x)$$

We denote $x_n \rightharpoonup x$.

Theorem 7.1.8: Weak compactness

From every bounded sequence in a Hilbert space, one can extract a weakly convergent subsequence.

Proof. Let (x_n) be a bounded sequence. If H is separable, let (e_k) be a Hilbert basis. By Cantor's diagonal argument, we extract a subsequence $(x_{\varphi(n)})$ such that for every k, $((e_k|x_{\varphi(n)}))$ converges. We then define a linear form on $\mathrm{Vect}\{e_k\}$ by:

$$L\left(\sum \alpha_k e_k\right) = \lim_{n\to\infty}\left(\sum \alpha_k e_k \middle| x_{\varphi(n)}\right)$$

and we extend it by continuity to H. By Riesz, there exists x such that $L(y) = (y|x)$ for every y. ■

§7.1.8 Application to L^2 Spaces

Theorem 7.1.9

The space $L^2(\Omega)$ equipped with the inner product:

$$(f|g) = \int_\Omega f(x)\overline{g(x)}dx$$

is a Hilbert space.

Proof. The completeness of L^2 is a consequence of the Riesz-Fischer theorem. If (f_n) is Cauchy in L^2, we extract a subsequence convergent almost everywhere, and the limit belongs to L^2. ■

Theorem 7.1.10: Hilbert basis of $L^2([0, 2\pi])$

The trigonometric system

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}\right\}_{n\geq 1}$$

is a Hilbert basis of $L^2([0, 2\pi])$.

§7.2 Exercices

Exercice 29. *Let $E = \mathcal{C}([0,\pi], \mathbb{R})$ equipped with the infinity norm $\|f\|_\infty = \sup_{t\in[0,\pi]} |f(t)|$. We fix a function $\varphi \in E$ and define the following operator:*

$$\begin{aligned} T : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^\pi f(t)\varphi(t)dt \end{aligned}$$

1. *Show that the operator T is a linear continuous operator.*

2. *Assume that $\forall t \in [0, \pi], \varphi(t) \geq 0$. Calculate $\|T\|$.*

3. *Set $\varphi(t) = \cos(t)$. Calculate $\|T\|$.*

Exercice 30. *Let H be a Hilbert space over $\mathbb{R}$ and $T : H \longrightarrow H$ be a linear application. We assume that*

$$\langle T(x), x\rangle \geq 0, \forall x \in H$$

(we say that T is positive). Show that T is continuous. (Hint: Show that its graph is closed).

Exercice 31. *Consider the Banach space $E = \mathcal{C}([0,1], \mathbb{R})$ equipped with the norm $\|\cdot\|_\infty$. Show that the following operator:*

$$\begin{aligned} T : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt, \end{aligned}$$

is a linear continuous operator and calculate its norm.

Exercice 32. *Consider the Banach space*

$$c_0 = \Big\{(x_n)_{n\geq 1} \subset \mathbb{C} : \lim_{n\to\infty} x_n = 0\Big\}$$

which is the space of complex sequences that tend to 0, equipped with the norm $\|x\|_\infty = Sup_{n\geq 1} |x_n|$ where $x = (x_n)_{n\geq 1} \in c_0$.

Let $a = (a_n)_{n\geq 1} \subset \mathbb{C}$ be a sequence that satisfies for all sequence $x = (x_n)_{n\geq 1} \in c_0$, the series

$$\sum_{n\geq 1} a_n x_n \text{ is convergent}$$

For all $N \geq 1$, we define the linear form

$$\begin{aligned} \Phi_N : c_0 &\longrightarrow \mathbb{C} \\ x &\longmapsto \sum_{n=1}^{N} a_n x_n. \end{aligned}$$

1. *Show that the form Φ_N is continuous and calculate its norm.*

2. *Show that the sequence $a \in \ell^1$, where*

$$\ell^1 = \left\{ x = (x_n)_{n\geq 1} \in \mathbb{C}; \|x\|_1 = \sum_{n\geq 1} |x_n| < \infty \right\}.$$

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